Relative Motion and Spacetime

Einstein's Theory of Special Relativity

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Preface

Why would a professional economist want to learn about Einstein's theory of Special Relativity? The answer is quite simple: curiosity. My focus on my career left me well educated in my chosen field but poorly educated in other areas. Perhaps my most significant lacuna was in the sciences.

So after I retired I decided to develop a better understanding of the world around me. I had long realized that for me the best way to learn was to teach, and the best way to teach was to write down my message in detailed lecture notes. In that way I would be forced to see more clearly the flaws in my understanding and, hopefully, I would communicate a clearer message. So I learned to learn by writing.

My education gave me the mathematical skills to apply this curiosity in the most fruitful manner: by going through the details of the theory and deriving the theory as a physicist might. Simply reading and remembering a list of conclusions was not a path to understanding: remembering a conclusion is, for me, more difficult than remembering the path to that conclusion.

So what follows is best seen as notes for an undergraduate lecture on Special Relativity, to be given in a hypothetical course to imaginary students. It opens with a brief description of space and time in classical mechanics, the field created by Isaac Newton in the 17^{th} century. Then it summarizes the basic conclusions of relativistic mechanics, created by Albert Einstein in the early 20^{th} century. The remaining sections provide the foundations for those conclusions. One who wants to jump to the conclusions need read no further than the second section.

One might ask, "Who cares?" The answer is that we who are rooted on Earth never directly notice the effects of relativity because we are all at rest with respect to each other, and because even if we are moving relative to one another, the motion is at such low speeds that we never see relativistic effects. But we are all constantly facing the effects of relativity in unseen ways. For example, pilots, mariners, and, lately, even drivers and walkers use Global Positioning Systems to navigate. Because the dozen or so GPS satellites are orbiting Earth at 14,000 km/h, the transmission of information between them and the dozen or so satellites must be corrected for relativity effects on time and distance; the effect is small but it is cumulative. Another example: physicists engaged in particle research involving high energies must translate the times and distances observed for particles between their—the physicists'—time and distance and the particles' time and distance. None of *us* have to know how this is done, but we are all affected by those who do know.

1. Important Distinctions in Relativity

While learning about Special Relativity (SR) I encountered many sources of confusion. These were not so much in the mathematics, but in the uses of language and in the interpretation and application of the theoretical concepts. Here I will list some of the more important areas of confusion.

Measuring "Time," "Distance," and "Mass"

The essence of Special Relativity is the distinction between time, distance, force, mass, energy and other physical attributes as seen by two or more observers when they are in uniform motion (constant velocity) relative to each other. For each observer there is a different measure of these attributes, so, for example, "time" might refer to: the stationary observer's measure of his own time using his own clock; a stationary observer's measure of a moving observer's time on the moving observer's clock; the moving observer's measure of his own time on his clock; or the moving observer's measure of the stationary observer's time on the stationary observer's clock.

To make this a bit less confusing, we use the term "proper" when referring to the moving observer's measurement of his own attributes. If M is an moving at constant velocity relative to S, a stationary observer, *proper time* is M's time as measured by his own clock, *proper distance* is the distance from M as measured by M's own ruler, *proper mass* is mass measured by M on M's own mass meter, and so on. The notation for these "proper" attributes is to attach an apostrophe to the symbol used: t' is the proper time,; x' is the proper distance, and so on.

The time on M's clock as measured by S will differ from proper time because of the effects of relative motion on S's perception of M's; this is true of most of S's perception of M's physical attributes. It is common to refer to S's perception of M's time as coordinate time, and similarly for all the other attributes. I call this simply "time", or "distance," or "mass," etc. These are denoted as t, x, and so on.

This language makes sense. When we on Earth talk about time, distance, and mass we are usually referring to the time, distance and mass on Earth—with which we are at rest: the time to drive from Boston to San Francisco, the distance from Boston to San Francisco, the mass of the automobile we are driving. To us, that is "time," "distance," and "mass." So we use "proper" to denote actual measurements on the moving object.

What about M's measurement of S's proper time, distance, and mass? Well, M is at rest relative to himself, and the measurements he makes on his instruments are his "time," "distance," and "mass" for his attributes. From M's perspective, S is moving and M's measurements of S's attributes are now the "proper" attributes. The theory holds that M's view and S's view are symmetric: if we take S's point of view, M (we will see) is seen as moving in slow motion; if we take M's view, S is seen as moving in slow motion. There is no such thing as "absolute" time, mass, etc.

Invariant Attributes

A second distinction is between attributes that are *invariant* and those that are not. Invariant attributes are measured as the same for all observers regardless of their relative motion. They are important because…well, because they are invariant: everyone sees them the same way. Invariant magnitudes are observer-*in*dependent. Under relative motion attributes that are not invariant are measured differently by every observer; they are observer-dependent.

The primary invariant magnitudes in Special Relativity are proper time, proper distance, proper mass (usually called *rest mass* or *invariant mass*), proper energy (also called *rest energy* or *invariant energy*), velocity, speed of light, and the *invariant interval*.

For example, we will see that distance is not invariant: S will measure M's distance from S as different from M's measurement of his distance to S. So using the term "distance" ignores the question "as measured by whom?" But we will see that there is an invariant measure of distance, called the *invariant interval*. The invariant interval is not a distance in space, because distance in space is not measured the same by all observers. Nor is it a distance in time, because that also is measured differently by different observers. The invariant interval is a shared measure of distance in *spacetime*. If S measures the invariant interval between two spacetime positions ("events") as " 23 light years," M will also measure that invariant interval 23 light-years.

Perception and Reality

In descriptions of Special Relativity one often comes across statements like "A stationary observer sees the clock of a moving observer as ticking slower." The language of "thought experiments," which we will discuss, is rife with these statements. Enough of this language makes you thinking that it is all just an illusion—that one clock is not *really* moving slower. In a sense, it is an illusion—if both clocks could be put side-by-side, they would tick at the same rate. But that is because they would be at rest with each other, not because a moving clock doesn't really tick slower than a stationary clock.

So reader beware: Special Relativity is not a mere illusion: relative motion does really affect physical attributes!

Units of Measurement

There are several measurement systems used to measure mass, distance, energy, force time, and so on. For example, the *Standard International System (SI)* uses the kilogram for mass, the meter for distance, and the second for time. The *CGS System* uses the gram for mass, the centimeter for distance and the second for time. The *Poundal System (PFS)* uses the pound for mass, the foot for distance, and the second for time; the PFS was common in England before Napoleon introduced the metric system. The modern convention, which we follow, is to use the SI system.

The derivation of measures of time, distance, and mass clearly depends on the units of measurement. So while the equations used in physics might look the same, they will give very wrong answers unless consistent units of measurement are used.

An appendix lists the important units of measurement and the conversions between them.

Box 1
Important Definitions

In what follows we use the definitions:

c: the speed of light is, say, meters per second
ct: the time in S's frame, measured as, say, light seconds, as seen by S (time)
x: the distance of an object or event from S's position, as seen by S (distance)
m: the mass of an object at rest with M, as seen by S (mass)
ct': the time in M's frame, as seen by M, in light-seconds (*proper time*)
x': the distance of an object or event from M, as seen by M (*proper distance*)
m'; The mass of an object at rest with M, as seen by M (*proper mass,* or *rest mass*)
β: the "rapidity" of a moving object relative to the speed of light; $\beta = v/c$
γ: the "Lorentz factor" (also "dilation factor"); $\gamma = 1/\sqrt{(1 - \beta^2)}$

2. The Classical View of Space *and* Time

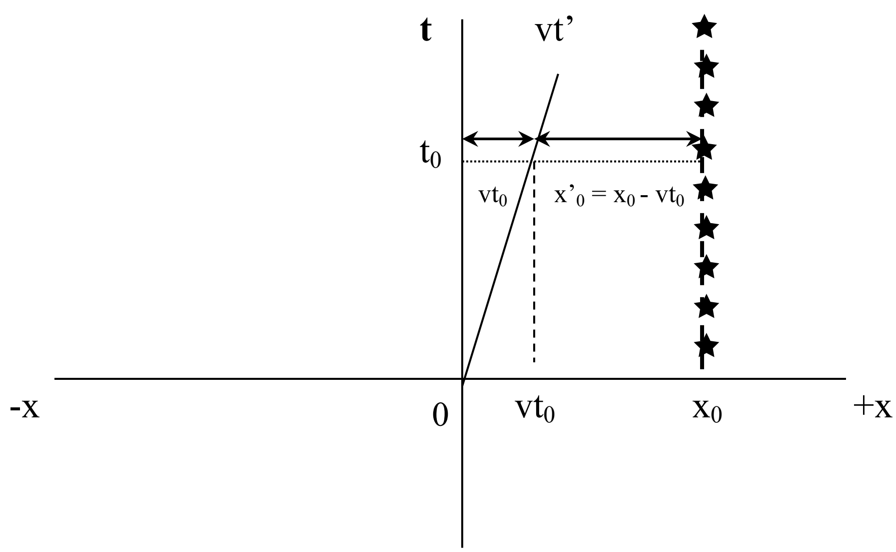
The classical view of space and time rests on Newton's Laws of Motion. Suppose that S is stationary and M is moving relative to S at a constant velocity v, which is less than the speed of light, c. Both S and M are said to have an inertial frame of reference for time and space, but one frame is moving relative to the other.

Suppose also that there is an object—a star—at a constant position in S's inertial frame. For simplicity, reduce "space" to one dimension—distance from the initial position: right ("away from S") or left ("toward S"), with S's initial position at the origin; up/down and in/out are ignored. Then we can describe the star in S's frame as being at the coordinates ($\mathbf{x_0}$, $\mathbf{t}$) in Figure 1: it is always at x_0 even as time changes, as in Figure 1.

The vertical axis (0, $\mathbf{t}$) is S's *timeline* (often called his *worldline*) because it shows S's path through time only (he is always at distance x = 0 from himself). The star's timeline is the vertical line at ($\mathbf{x_0}$, $\mathbf{t}$) because at all times the star stays at distance x_0 from S.

S's inertial frame is described by the Cartesian coordinate system, as shown below (*for the moment, ignore the ray vt'*).

Figure 1
Classical Distance and Time



Now suppose that an object—a spaceship with M aboard—takes off from the origin and heads toward the star (rightward) at velocity v. Let t' represent the spaceship's proper time, and x' represent the spaceship's proper distance from S, both as measured by M. The timeline for the spaceship is the ray vt' running out of the origin—it shows all the positions of the spaceship in S's frame as time increases.

How far is the spaceship from the star at S-time t_0? The Newtonian answer is $(x_0 – vt_0)$, the distance from S to the star less the distance traveled by M toward the star. And M measures the same time and distance; he has proper distance $x' = (x_0 – vt_0)$ to go, and he has traveled for $t' = t$. Time and distance are invariant in the Newtonian system: for both S and M time and proper time are the same, as are distance and proper distance.

Box 2
The Classical Transformation of Motion

$$x' = x – vt$$
$$t' = t$$

This is the way we normally think about time and distance. We all see it the same way: a New York City resident measures the distance to Paris as the same number of kilometers that a Paris resident measures to New York. And at 11:00am GMT in New York it is also 11:00am GMT in Paris. Time and distance are invariant attributes.

But Einstein found that the classical translation does not apply when the relative velocity of the moving observer motion is "high." To address this, Einstein united space and time into *spacetime*, and found that the normal physical attributes are *not* invariant.

3. Einstein's View of *Spacetime*

Special Relativity requires only two postulates:

1. The speed of light is the same for all observers: each observer sees the speed of light as c *regardless of the relative motion of the light source or of the observer*. If a flashbulb goes off on earth at the same time as a spaceship passes earth going 100,000 km/sec, both an earthbound observer and a spaceship observer will measure the speed of that light as 300,000 km/sec. In contrast, the classical answer would be that light travels at 300,000 km/sec relative to the Earthbound observer, but at only 200,000 km/sec relative to the spaceship observer.

2. All laws of physics are the same in both a stationary observer's frame and a moving observer's frame. For example, Newton's Second Law of Thermodynamics, law F = ma (Force = mass times acceleration), applies in both frames. Force and Mass might be measured differently, as we will see, but the same Law applies.

From these two postulates Einstein concluded that Newtonian dynamics applies only when the spaceship is traveling at a low speed relative to the speed of light. At very high velocities, S's perception of M's spacetime coordinates is "warped." Time and space are not "really" different for S and M: each has a Cartesian view of their own space and each measures time in seconds on their clock. It is S's perception of M's time-space position that is altered by M's relative motion. This effect is symmetric: In M's frame his spaceship is in a Cartesian space-time, but M's perception of S's frame is warped!

Before getting into the mathematics of Special Relativity, we first highlight some of the important implications. The reader who wants only the conclusions can stop at the end of section 3.

3.1 The Relativity of Time and Distance

In Special Relativity we are comparing the spacetime of a stationary observer, S, with the spacetime of a moving observer, M. Unlike the classical framework, Special Relativity sees time and space as inextricably related: each observer will measure time and distance differently.

Let (x, ct) be the spacetime coordinates of S, the stationary observer, and (x', ct') be the spacetime coordinates of M, the moving observer, as seen by M. According to Special Relativity:

$$t\,/'t = 1/\sqrt{(1 - \beta^2)}$$

and

$$x/x' = 1/\sqrt{(1 - \beta^2)}$$

where β = v/c is the rapidity of the moving object and γ is the Lorentz factor (see Box 1). As v goes from zero to c, proper times and distances don't change but time and distance (t and x), as measured by the stationary observer goes to infinity. Thus, S's measurement of M's time and distance dilates, or increases, relative to M's on-board measurements.

For example, suppose at t = t' = 0, M leaves Earth on a spaceship heading to a distant star, as in Figure 1. Suppose further that M's clock says that M has been underway for 2 light years (t' = 2) and that there are 10 light years to go before the star is reached (x' = 10). If the spaceship's velocity is ½ of light speed (β = ½) then t/t' = x/x' = 1.15. ***S thinks M has been underway for 2.3 light-years and has 11.5 light-years to go to the spaceship's destination***.

Table 1
Dilation and Rapidity

β	t/t'	t't
.75	1.51	0.66
.90	2.29	0.44
.95	3.20	0.31
.99	7.09	0.14
.9999	70.71	0.01
1.0000	∝	0.00

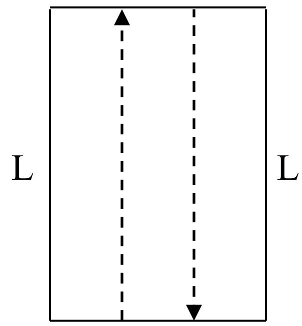
The clock on a spaceship moving at 99% of the speed of light will be seen by S to have time moving at only 14% of the rate of S's clock! And at 99.99% of light's speed t'/t = .01! As v approaches c (β approaches 1), t explodes and t' vanishes.

Note that the dilation of distance is only in the spaceship's direction of travel! If we convert to 4-space (3 space dimensions and one time dimension) we find dilation only in the t and x directions; directions perpendicular to the direction of travel (y and z) show no dilation and both S and M would agree on transverse motion.

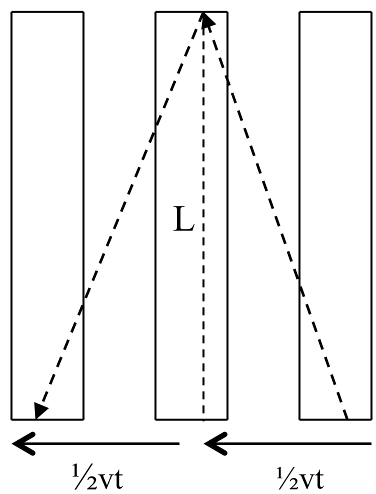
<u>A Thought Experiment</u>

Einstein often approached complex problems with thought experiments: imagining physical analogues to get simple insights into the answers that would require more complex analysis to nail down. One example is *Einstein's Light Clock*, used to demonstrate time dilation.

A light clock is a hypothetical device to measure time precisely. It is a box with a mirrored bottom and top. A photon is emitted from the bottom and bounces off the top and returns to the bottom. The round-trip time is a unit of time—one tick of the clock. Suppose that M's ship is carrying the light clock shown below.



To M, an observer on the light clock, light travels a distance of 2L in one time-unit. Because light travels at velocity c, a time unit is $t' = 2L/c$.

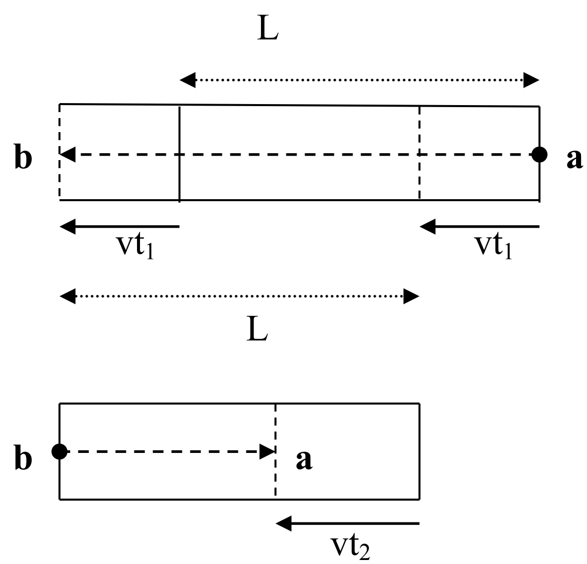


But to S, an observer sitting in a fixed position outside the light clock, M's light clock moves leftward relative to S at velocity v. The (proper) time on the light clock will be $t' = 2L/c$ because it is "at rest" with itself. But to S the clock is moving and the photon will be seen as following the path shown above.

The distance traveled by the photon (as seen by the stationary observer) on the upward path is, by the Pythagorean Theorem, $\sqrt{[L^2 + (\tfrac{1}{2}vt)^2]}$. The down-path distance is the same, so the total distance traveled is $2\sqrt{[L^2 + (\tfrac{1}{2}vt)^2]}$ and the time taken for the round-trip is $t = (2/c)\sqrt{[L^2 + (\tfrac{1}{2}vt)^2]}$. This can be written as $t = \sqrt{[t'^2 + \beta^2 t^2]}$, so ***$t = t'/\sqrt{(1 - \beta^2)}$***. *S's measure of time is M's proper time dilated by the effects of motion.*

Length Contraction with a Light Clock

Suppose that the light clock is laid on its side, and that it moves leftward at velocity v. Consider the picture below. In the time (t_1) that it takes for the photon to be emitted from one end (point **a**) and received at the other end (point **b**) the clock has moved distance L + vt_1.



The photon is then reflected back from point **b** but it only travels L - vt_2 in the time before it is received at point **a**; the reason is that the return trip is shorter because the clock's detector has moved another vt_2 units to the left during the return trip.

The simple approach to seeing length dilation is to note that the round trip distance **a**-**b**-**a** on the clock as measured by M, an observer on the clock, is proper distance 2L' at velocity c in proper time t' = 2L'/c. But to a stationary observer the photon covers a round trip distance of 2L at velocity c in time t = 2L/c.

Now, from time dilation we have

$$t = t'/\sqrt{(1 - \beta^2)}$$

from which, since t = 2L/c and t' = 2L'/c, we get

$$L = L'/\sqrt{(1 - \beta^2)}$$

A more detailed story is that in S's frame the time taken for the round trip is $t = L'/(c-v) + L'/(c+v)$, or $t = 2(L'/c)/(1 - \beta^2)$. Because time dilation gives $t = t'/\sqrt{(1 - \beta^2)}$ we have $t' = 2(L'/c)/\sqrt{(1 - \beta^2)}$. But t' = 2L'/c, so $\boldsymbol{L = L'/\sqrt{(1 - \beta^2)}}$.

Box 3
Implications of Time and Space Dilation

- In the limit, if M is moving at the speed of light, M 's time will stop relative to S's time. If M is on a spaceship being drawn into a black hole at the speed of light, a distant observer will see the spaceship as frozen in time and space even though M sees himself moving at lightspeed.
- A muon is a very short-lived particle, created by cosmic rays hitting atoms in Earth's upper atmosphere. Muons appear and disappear in 0.000002 (two Millionths) of a second, during which they travel about 0.372 miles. Yet we can observe the muon in detectors on the Earth's surface! The reason is that the muon moves at close to the speed of light, so a .000002 seconds in its proper time translates to an observable interval of time on our clock. If a muon moves at 0.9999c, it will exist for 0.000141 seconds Earth-time and travel a distance of 26.3 miles in Earth-distance.
- The faster M is traveling, the shorter M's spaceship will seem to S. In the limit, if M is traveling in a spaceship moving at the speed of light, the length of M's spaceship will be seen by S as zero.
- A lightbeam leaves Alpha Centauri. In Earthtime (t) it takes 4 billion years to reach earth. But in its own time (t') it reaches earth instantaneously.

3.2 The Relativity of Mass

Let m' refer to proper mass—the mass of an object traveling with the spaceship as seen by its pilot, and m refer to *relativistic mass*—the mass of that object as seen by S. Just as time and distance are dilated by relative motion, so also is the object's mass. Mass is defined in several different ways: because momentum is mass times velocity (p = m'v), mass is momentum divided by velocity (m' = p/v); because force is mass times acceleration (F = ma), mass is force divided by acceleration (F = F/a). So if one knows momentum and velocity, or force and acceleration, mass can be calculated. [Note that force is the rate of change of momentum with respect to proper time (F = dp/dt' = m'a)].

For an object moving slowly relative to the speed of light, the force required to achieve acceleration rate **a** is m'**a**. But the higher the velocity of the object, the greater the force required to maintain acceleration rate **a**, and as v approaches close to c, the force required becomes infinite. The reason, we shall see later, is that at high speeds proper mass is unchanged but mass (m), as seen by S, become infinite as v approaches c. For this reason, c is the *cosmic speed limit*: nothing can move faster because at v = c mass is infinite and any acceleration would require infinite force.

The relativity of mass was postulated by Einstein as a result of the Law of Conservation of Momentum; it has withstood numerous experiments. This development will be discussed later.

If m' is the *rest mass* of an object traveling with M, as measured by M, then we find that the *relativistic mass*, m, as measured by S is m = γm'. Later we return to a deeper explanation of relativistic mass.

Box 4
Mass Dilation

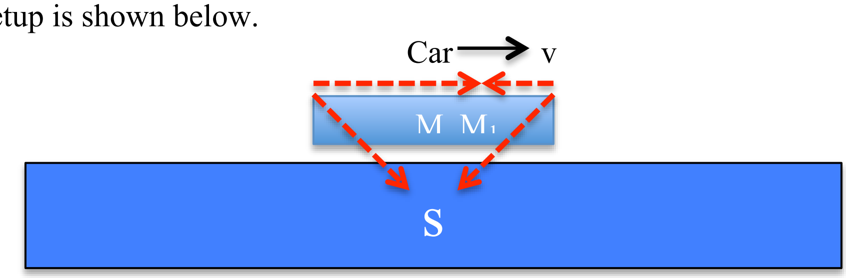
$$\mathbf{m = \gamma m'}$$

- The Large Hadron Collider in Zurich accelerates protons to 99.9999 percent of the speed of light. If m' is the rest mass of a proton, the relativistic mass of the proton is about 70m': an observer outside the LHC sees the mass as increased 70-fold.
- An asteroid approaches the earth at one-tenth of light speed. The asteroid's rest mass is m'. The relativistic mass of the asteroid when it hits the earth is 1.005m'. If the object is moving at 95% of lightspeed (β = .95) its mass is 3.2 times its rest mass. As velocity approaches the speed of light, relativistic mass becomes infinite.

3.3 The Relativity of Simultaneity

We are accustomed to the idea that if we see two events as simultaneous (or one as preceding the other), every other observer will agree to that ordering. But Special Relativity argues that this is a classical notion that breaks down when there is relative motion.

Einstein used a thought experiment to give a simple explanation of how simultaneity breaks down when there is relative motion. Suppose observer S is standing on a platform as a train goes by at velocity v. One of the cars is rigged with a light at each end. Just when the midpoint of the car passes S the conductor turns both lights on. At that very moment a passenger M is sitting in the middle of the moving car exactly across from S. The setup is shown below.



S will see the first photons from each light at precisely the same moment because when the lights came on the distance from each light to S is the same, and because light travels at speed c regardless of the motion of the source; so to S they came on simultaneously.

But M does not see the lights as coming on simultaneously; he sees the forward lights coming on first because while the first photons from each light had been coming toward him at velocity c, he has moved rightward with the car to M_1 so photons from the forward light have a shorter distance to travel before reaching M's eyes.

So Einstein has shown that two observers will not necessarily agree on whether events are simultaneous if one observer is moving relative to the other.

3.4 The Twin Paradox: Symmetry of Spacetime Translations

We have seen that S measures the space and time of a moving object differently than does an observer on the object: both time and distance are expanded when translating from M's frame to S's frame. This says that S will view M as moving in slow motion and that S sees himself as aging faster than M.

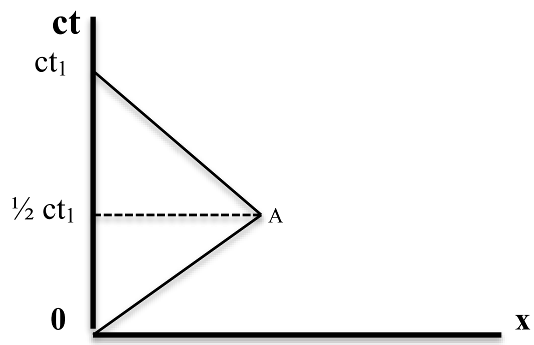
What about the view from M's frame? It is identical: M thinks he is stationary and that S is speeding away from M. So M will view S as in slow motion, and believe that he, M, is aging faster than S.

So both see the other as aging slower. And each will measure the other's distances as shorter. Which is right? The answer is—both! Or neither! Well, you can't "really" say unless you can directly compare ages: time is relative!

This apparent inconsistency gives rise to the famous Twin Paradox. Two twins on earth are given different tasks. One is to stay on earth while the other departs rightward in a spaceship at rapidity β. Each has an instantaneous view of the clock at the other's position (actually, that is impossible because of the time it takes for a signal to travel between S and M; but we assume it is possible). Each records the readings of the other's clocks and determines relative ages. We know that each will record the other as aging slower.

At some point M's spaceship reverses direction and heads directly back to earth at the same rapidity. The path taken is shown in Figure 2 below: From $t = 0$ to $t = ½\, t_{1,}$ M's ship is outbound on segment 0A. At A it reverses direction (we ignore the need to decelerate, turn, and accelerate back to the initial velocity) and a $t = ct_1$ the twins are reunited.

Figure 2
The Twin Paradox Path



Both will have recorded the other as younger. But is one really younger? The answer is that M, the spaceship twin, is younger than S, the earthbound twin, and the age difference increases with higher rapidity. It is possible that M returns to Earth to find that his twin is long dead. At the extreme, if the spaceship is going at light speed, M will not have aged a bit while S will be quite a bit older, or, perhaps, has crumbled to dust.

We will return to this paradox when we have established the mathematical foundations of Special Relativity.

4. The Mathematics of Special Relativity

4.1 The Lorentz Transformation

The mathematical basis of Special Relativity was created before Einstein but its implications were overlooked. In the late 19^{th} century Hendrik Lorentz, a Dutch physicist, postulated that space and time are transformed by motion. The *Lorentz Transfomation* addresses the translation of one coordinate system into the coordinate system moving relative to the first system.

Observer S is in an inertial frame (x, ct) and observer M, moving at velocity v in the x direction relative to S, is in another inertial frame (x', ct'). Both distance (x or x') and time (ct or ct') are measured in equivalent units—light-seconds.

The Lorentz transformation describes the relationship between the two frames as

$$\text{(1a)} \quad x' = \gamma(x - \beta ct)$$

and

$$\text{(1b)} \quad ct' = \gamma\,[-\beta x + ct]$$

for M's proper time and distance as functions of S's time and distance. Inverting the relationships, S's perception of M's proper time and distance are

$$\text{(2a)} \quad x = \gamma\,(x' + \beta ct')$$

and

$$\text{(2b)} \quad ct = \gamma\,[\beta x' + ct']$$

If there is no relative motion the classical transformation applies: when $v = 0$, then $\gamma = 1$ and the transformation from S's perspective to M's frame is the classical Newtonian transformation described in the second section:

$$\text{(3a)} \quad x' = x - vt$$

and

$$\text{(3b)} \quad t' = t$$

But when rapidity is high, the Newtonian transformation fails.

A strange thing happens when $\beta = 1$ Noting that light speed in both frames is equal, i.e., $x/t = x'/t = c$, we get

$$\text{(1a)} \quad x' = \gamma(x - ct) = 0$$

and

$$\text{(1b)} \quad ct' = \gamma\,[-x + ct] = 0$$

That is, when M is traveling at the speed of light he sees all distances as zero (he is "everwhere") and all times as zero (he is "everywhen"). Because only photons and other massless particles can travel at light speed, a photon leaving Alpha Centauri arrives instantaneously at Earth (because its distance from Earth is zero) even though we see it as traveling for four light-years! This is a very perplexing, but correct, idea.

Box 5
Time and Distance Dilation

$$t'/t = 1/\sqrt{(1 - \beta^2)}$$

$$x/x' = 1/\sqrt{(1 - \beta^2)}$$

4.2 Minkowski Diagrams and Minkowski Space

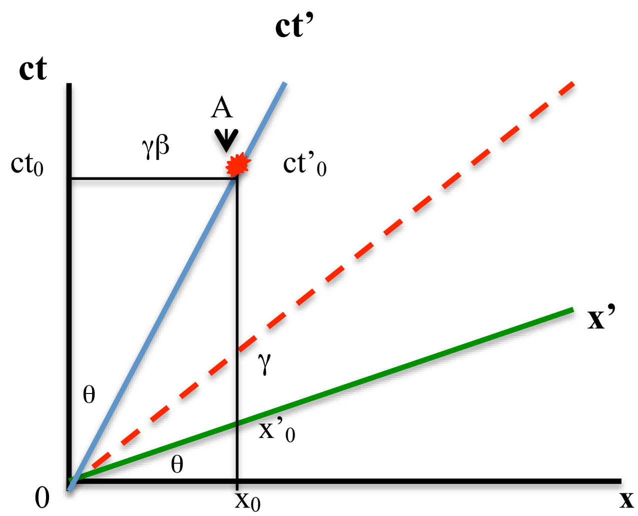
Hermann Minkowski was one of Einstein's professors at the Swiss Federal Institute of Technology in Zurich. In 1908, following Einstein's 1905 paper on Special Relativity, Minkowski wrote a paper elucidating the new theory by describing the spacetime geometry implied by Special Relativity. The result has become enshrined in the *Minkowski Diagram* and in the notion of *Minkowski Space*.

Under a Lorentz Transformation the coordinate system for S is the familiar Cartesian system (x, ct) shown below in Figure 3: S's position is x light-seconds from the origin, and his time is ct light-seconds from the initial time; both are chosen to be zero at the outset (Note that a light-second is used as both a unit of time and a unit of distance). The dashed red line in the diagram below represents a light ray from the origin; it is at a 45º angle (slope = 1) because both axes are measured in the same units, light-seconds: any point on the lightline represents velocity v =c, at which x/t = c or ct =x.

The coordinate system of a moving observer, M is (x', ct'); it is described above by equations (1a) and (1b). This is represented in Figure 3 by a rotation of the time and space axes, as shown by the blue and green rays. The blue line is the moving object's *timeline*, showing all of M's possible positions in time when he is at distance x' = 0; the green line is his *spaceline*, showing all of M's possible positions in space when he is at t' = 0. Both lines start at the origin because we assume a starting position of t = t' = 0 and x = x' = 0.

The effect of relative motion is to rotate M's timeline clockwise from S's by angle θ, and to rotate M's spaceline counterclockwise by the same angle; that angle of rotation is θ = arctan(β)—it increases as v rises, reaching 45º when v = c, i.e., when β = 1. When M is moving at light speed both his timeline and his spaceline will coincide with the lightline because their slopes approach 1 as β approaches 1. In this case M is "everwhere" and "everywhen."

Figure 3
Minkowski Space



For example, at point (x_0, ct_0) in S-space, M is at (ct'_0, x'_0): ct'_0 is the distance from 0 to ct'_0 on the blue ray; this is not equal to the distance from 0 to ct_0 on the vertical axis, as we will soon see. Similarly, the distance from 0 to x'_0 on the green ray is not equal to x_0 on the horizontal axis. Thus, relative motion changes the units of time and distance measurement.

What is the relationship between a unit of S's spacetime and a unit of M's spacetime? How many "inches" along M's timeline are equivalent to one "inch" along S's timeline?

Consider the event at A, that is, at (x_0, ct_0) in S-Space. Suppose that event A is at $(ct'_0, x'_0) = (0, 1)$ in M's coordinate system. From the Lorentz Transformations $x = \gamma\,(x' + \beta ct')$ and $ct = \gamma\,(\beta x' + ct')$ we see that the point (0, 1) on M's timeline and spaceline maps to $(\gamma\beta, \gamma)$ on S's timeline. Therefore (x_0, ct_0) on S's timeline is equivalent to $(\gamma\beta, \gamma)$ on M's timeline.

Using the Pythagorean Theorem we find that the distance 0A on ct' (M's timeline) is $ct' = \gamma\sqrt{1 + \beta^2} = \sqrt{(1 + \beta^2)}/\sqrt{(1 - \beta^2)}$. Thus,

$$\text{one unit } t' = \sqrt{(1 + \beta^2)}/\sqrt{(1 - \beta^2)} \text{ units of } t$$

It can be shown that the same applies to distance, that is

$$\text{one unit } x' = \sqrt{(1 + \beta^2)}/\sqrt{(1 - \beta^2)} \text{ units of } x$$

So, to measure M's distance using S's distance and time on a Minkowsi Diagram, we use the transformations in Box 6:

Box 6
Mapping S-Space to M-Space

one unit t' = $\sqrt{(1 + \beta^2)} / \sqrt{(1 - \beta^2)}$ units of t

one unit x' = $\sqrt{(1 + \beta^2)} / \sqrt{(1 - \beta^2)}$ units of x

For example, suppose that A is at one inch along S's timeline and 1.5 inches on S's spaceline. If $\beta = 0.95$ the distances along M's time and space lines will be 4.44 times the distances to A along S's axes. [Note that this translation refers to distances in the Minkowski diagram: one "inch" on M's timeline is $\sqrt{(1 + \beta^2)} / \sqrt{(1 - \beta^2)}$ inches on S's timeline. It does not refer to the amount of time or distance dilation—that is measured by $\gamma = 1/\sqrt{(1 - \beta^2)}$.]

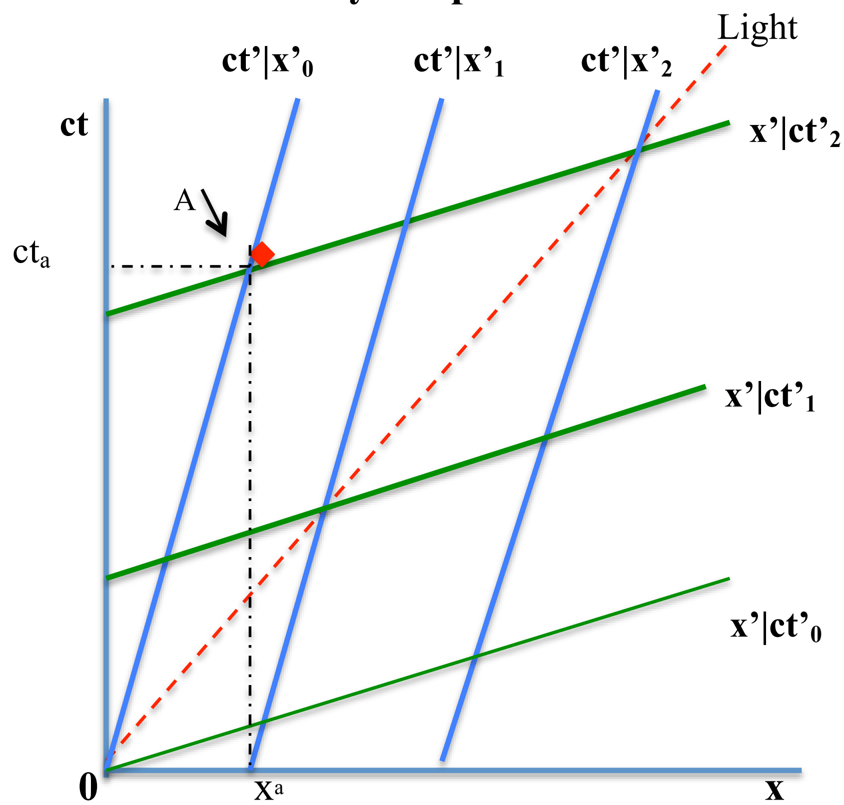
4.3 The Geometry of Minkowski Space

What are the numerical values of (x'_0, t'_0) associated with (x_0, t_0)? To answer this we have to fill in more of the family of M-space timelines and spacelines, as in Figure 4. Figure 4 shows the geometry of M's spacetime as seen by S (only the first quadrant is shown). Each green line, denoted as $x'|ct'_n$, is a spaceline showing all possible values of x' associated with a single value of ct'_n; thus, the green line labeled $x'|ct'_2$ is all values of x' that can occur when $t' = t'_2$. Each blue line, denoted as $ct'|x'_n$, is a timeline showing all the possible values of ct' associated with a single value of x'; the blue line labeled $ct'|x'_2$ is all values of ct' that can occur when $x' = x'_2$.

The dashed red line is the timeline for a light ray starting at **0**, where t = x = 0. This is also the point where the lines for x'_0 and ct'_0 intersect. Note that (x'_0, ct'_0), (x'_1, ct'_1), and (x'_2, ct'_2) are all points on this *lightline*. Note that the timeline and spaceline that intersect on the lightline all have the same angle with the lightline. This (and manipulation of the associated Lorentz equations) tells us that for all points on the lightline we have $x'/t' = x/t = c$, that is, the velocity of light is the same for both S and M; c is an invariant magnitude. Recall that this was one of Einstein's two postulates.

Any event in spacetime can be translated between S-space and M-space using the algebra underlying Figure 3. Thus, point **A** is an event at (x_a, t_a) in S-space. How do we visualize that position in Minkowski space? We can't do it using the Euclidean world of right angles because M-space is not Euclidean; it is "twisted." So here's how its done. Find the spaceline x'/ct' for M that passes through the event A: this, we see, is spaceline $x'|\, ct'_2$, so we know that t'_3 is the M-time. Follow that spaceline northwest to the A (at the red dot), where that spaceline intersects the lightline. Now find the timeline that passes through the same point on the lightline: it is ct'/x'_2, so we know that the event A occurs at (x'_2, t'_2) in M-space.

Figure 4
The Geometry of Spacetime



4.4 The Invariance of the Speed of light

Recall that Special Relativity postulates that the speed of light is c for all frames. In order for the speed of light to be the same for both S and M, it must be true that distance traveled divided by time taken equals the speed of light in each frame, i.e., x'/t' = x/t = c. We have seen that Special Relativity implies $x' = x\sqrt{(1 - \beta^2)}$ and $t' = t\sqrt{(1 - \beta^2)}$. Thus, since v = x/ct and v' = x'/ct' we get

$$x'/ct' = x\sqrt{(1 - \beta^2)}/ct\sqrt{(1 - \beta^2)} = x/ct$$

The speed of light is the same for both observers.

This can also be seen in the Minkowski diagram: time and spacelines always intersect on the lightline, which bisects the time and space lines.

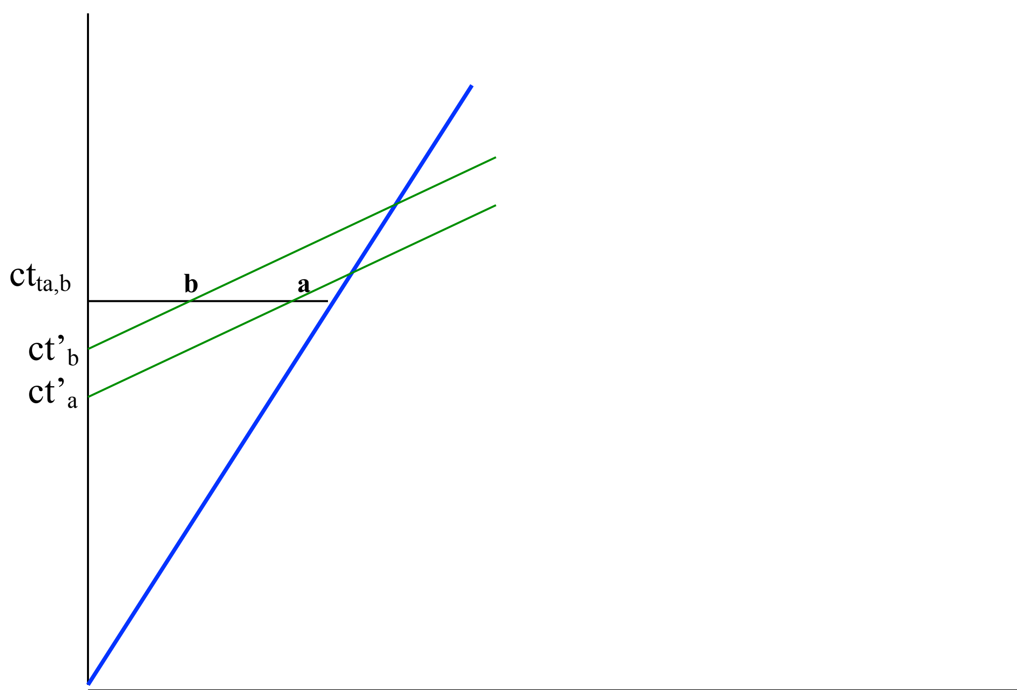
4.5 The Relativity of Simultaneity and Causality

Stationary observer S is inclined to say that if two events are revealed to him at exactly the same time, they are simultaneous. He also might believe that if event A is observed before event B, B could not have caused A because to do so would be to send information back in time, a Special Relativity no-no; he might also argue that if A precedes B, A caused B (though this is not necessarily the case because A could have independently occurred before B, and because the fact that A occurs before B does not mean that B was caused A.

But Einstein's theory shows that "simultaneity" is a relativistic concept: what appears as simultaneous in S's frame will not generally be viewed as simultaneous in M's frame which is moving relative to S. Also, if S sees A as preceding B, M might see B as preceding A. In other words, there is no absolute simultaneity or order-of-events!

Suppose that S sees two light bulbs go off at different distances but at the same time. S says that the two flashes are simultaneous. But S sees M as observing that one goes off before the other. Consider the Minkowski Diagram below.

Figure 5
Simultaneity in Minkowski Space



At time $t_{a,b}$ S sees a flash of light "simultaneously" at points **a** and **b**. But M sees flash **a** at time t'_a, before the flash at **b** at time t'_b. Simultaneity for S is not simultaneity for

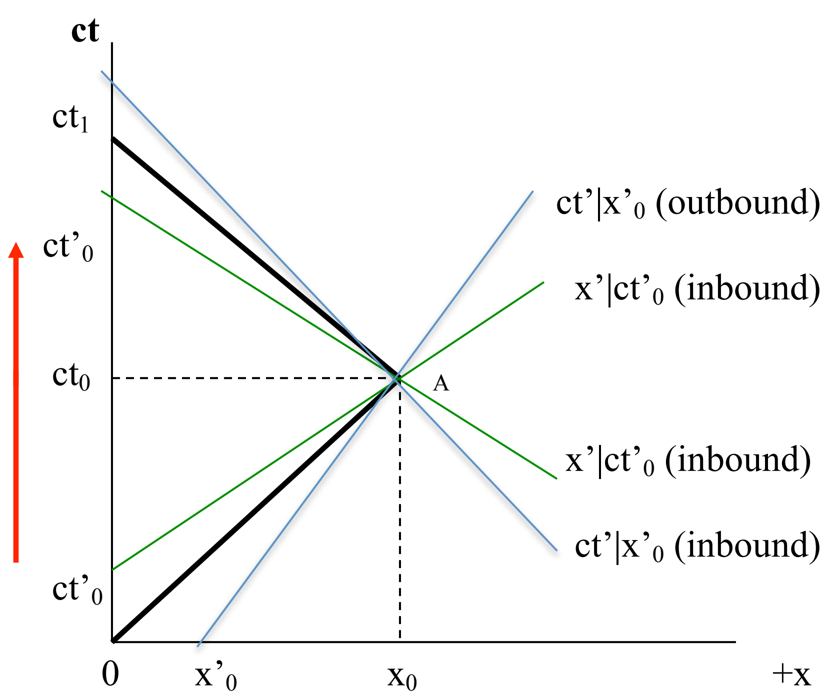
4.6 The Twin Paradox Revisited

Suppose that M, the twin on the spaceship, travels forever relative to his twin S. Which observer, S or M, ages more slowly? The question is meaningless: S sees M's clock as slower, but M sees S's clock as slower. Since the two will never come together, comparison of ages to find "the truth" is not possible.

But now suppose that S stays on Earth while M departs in a spaceship in the rightward direction at velocity +v. After a few years of travel, the spaceship reverses direction and returns to Earth at velocity -v.

Now the twins will come together. Which will actually have the grayer hair at that time?

Figure 6
The Twin Paradox in Minkowski Space



In Figure 6 M departs from Earth on a spaceship traveling rightward at velocity v along the heavy black line; at Earth time ct_0 M is x_0 light-years from Earth—at event A. M's ship time is ct'_0 which sets the upward-sloping green spaceline labeled "$x'|ct'_0$ (outbound)" and the blue timeline labeled $ct'|x'_0$ (outbound) that are in effect when the ship reverses direction at proper time at ct'_0 and proper distance x'_0.

At the moment of reversal the entire frame of Minkowski space changes dramatically. The spaceline and timeline at that moment when the ship begins on it return journey flip to the lines labeled "inbound" and the origin is set at A.

The proper time ct'_0 is shown twice on the vertical axis: the lower ct'_0 is the proper time the instant before M turns, and the upper ct'_0 is the same proper time but when the ship starts its return flight. Of course, ct'_0 is a fraction of S's time at the moment of reversal ct_0; the fraction is the by-now-well-known $\sqrt{1 - \beta^2}$.

So at proper time ct'_0 (coordinate time ct0) the ship reverses direction and returns to Earth at the velocity -v. The time of the ship's return to Earth is Earthtime ct_1. At that time, the Earth and ship clocks can be directly compared. The spaceship clock will have moved more slowly: it will be $2ct'_0$ while the Earthtime will be ct_1. There will be a significant segment of Earthtime—denoted by the heavy red arrow on the left—that is simply "skipped" as if the mere reversal of the ship's direction has made earthtime jump forward.

The moving twin will have aged more slowly! For example, if the spaceship is traveling at $\beta = .95$, elapsed proper time for the entire trip is 62% of elapsed earthtime. The remaining 38% of earthtime will have been "lost in time" when the ship reversed direction! If the outbound and inbound trips are each 10 years in earthtime, the Earthbound twin will be 20 years older but his astronaut twin returns only 12.4 years older.

What has happened? Well, it all has to do with the ship's acceleration when it reversed direction (acceleration is not simply change in speed, it is also change in direction). The shape of spacetime has altered with the ship's reversal and the new spacelines (represented by the inbound lines) have come into play. So Special Relativity implies that accelerating motion has a different effect on spacetime than does uniform motion. This would not be fully understood until Einstein introduced General Relativity in 1915.

Is the "lost time" really lost? Does S's record of time suddenly skip through the period of M's reversal, like a needle on a cracked phonograph record? Clearly not—S experiences every moment of the "lost time." But he sees the ship's reversal as taking up all of that time—in his view the ship just stops and very *very* slowly turns during the "lost time," then it begins its return to Earth.

5. Hyperbolic Spacetime: An Alternative View

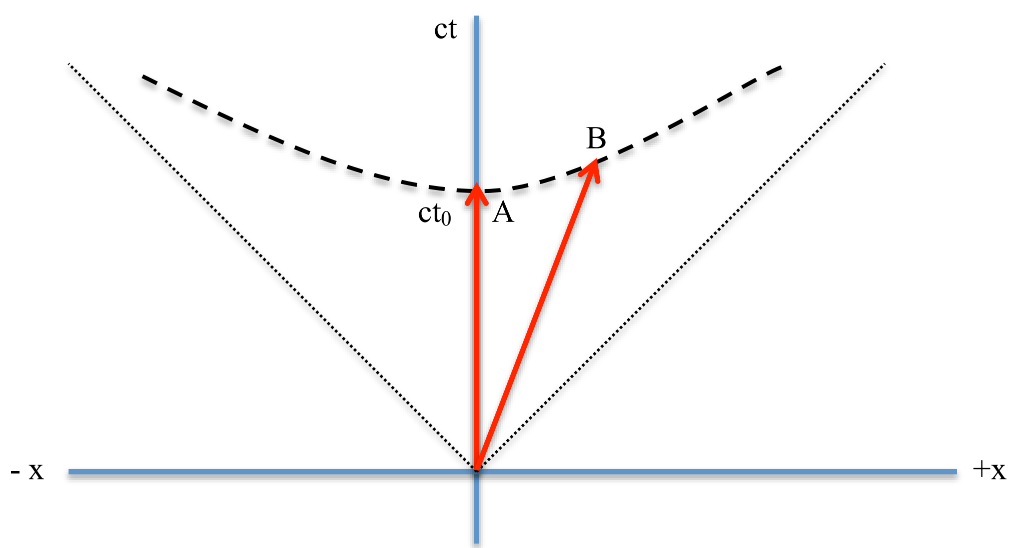
The theory of Special Relativity appears to reject any concept of absolute time or space that is invariant to the relative motion of observers. But there are attributes that are invariant. One absolute concept embedded in special relativity is the *invariant interval*, also called the *spacetime interval*.

5.1 The Invariant Interval

The invariant interval is the distance between events in spacetime. Denoted as s, it is defined as $s^2 = (ct)^2 - x^2$ for S's observation of his spacetime, and $s'^2 = (ct')^2 - x'^2$ for M's observation of his (M's) spacetime. While S and M can't agree on time and distance as separate features of spacetime, they do agree on the invariant interval in spacetime.

The invariant interval describes the geometry of spacetime as hyperbolic rather than Euclidean. The figure below shows this. The 45 lines show travel at light speed. The vertical red line is S's timeline, while the canted red line is M's timeline; both end at the bowed line representing the hyperbola associated with a specific spacetime (s); this is one of an infinite number of hyperboles, one for each possible interval.

Figure 7
Hyperbolic Spacetime and an Invariant Interval



Thus, at A and B we have both S and M agreeing that they are at the same spacetime. Because x = 0 for S on his timeline, he calculates the invariant interval at A as $(s_A)^2 = (ct_A)^2$; for M at point B, the invariant interval is $(s'_B)^2 = (ct'_B)^2 - \mathbf{(x'_B)^2}$. Both points are on the same hyperbolic line so $s_A = s'_B$. Note that this equality generates a familiar result: If $s_A = s'_B$ then $t' = t\sqrt{1 - \beta^2}$. We are back to time dilation via a different route!

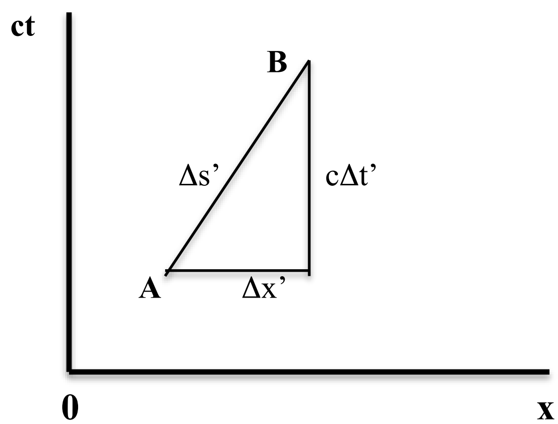
Note that At events A and B both observers agree on their position in spacetime, but S (who is at A) attributes all his travel to movement through time and he attributes M's motion to travel through both space and time. Because M has traveled more in space than S, and because the spacetime interval is the same for both, M has necessarily traveled less in time than S—time dilation again.

This raises an interesting question: does the speed of travel through spacetime depend on the time versus space breakdown of that travel? Is travel through one dimension slower than travel through the other?

5.2 Velocity of Motion in Spacetime

Consider the following figure, in which M travels through spacetime from **A** to **B**. The invariant interval traveled, Δs, is composed of two parts: cΔt is travel through time and Δx is travel through space; this is because $(\Delta s)^2 = (c\Delta t)^2 - (\Delta x)^2$.

Figure 8
Travel in Spacetime



Any measure of speed travelling through spacetime requires a measure of distance and a measure of time taken to travel that distance. Both must be invariant: the distance and time measures should be agreed on by all observers, as should the measure of velocity.

An invariant measure of spacetime time is Δs'/c: Δs' is M's spacetime distance and c is distance per unit of time, so Δs'/c is in units of time. An invariant measure of velocity is Δs'/(Δs'/c): Δs' is distance in spacetime and Δs'/c is time. But Δs'/(Δs'/c) = c: M's velocity as he travels through spacetime is c, the speed of light! S travels only in time, so he also travels at velocity c. *All objects traveling through spacetime move at the speed of light!*

But the breakdown between travel through time and through space will be different for every observer: M is moving through both space and time, so if he travels through the same invariant interval, also at velocity c, he must be traveling through a shorter time to make up for his movement through space. Time dilation again again!

5.3 Classifications of Spacetime

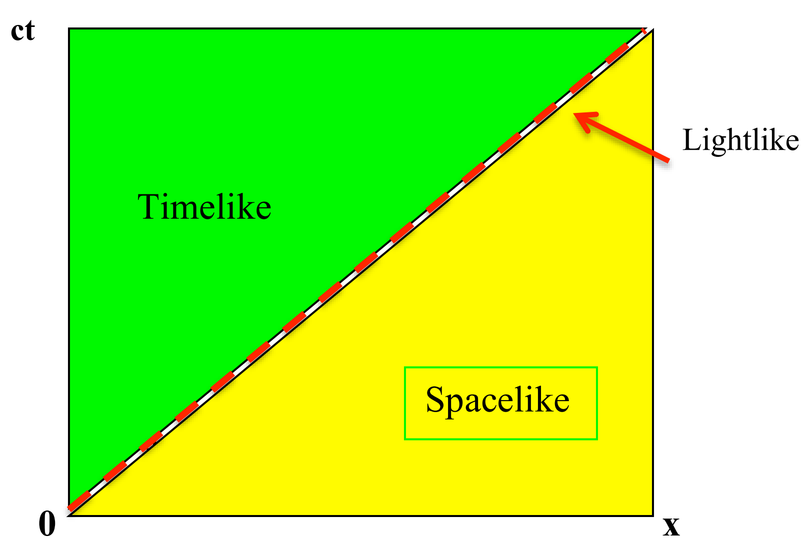
The invariant interval defines three classes of events:

1. *Timelike Separation*: if $s^2 > 0$ then $x/t < c$ and $x'/t' < c$: Events with timelike separation occurs at a position with velocity less than the speed of light. This means that these events fall into the zone above the light-ray in a Minkowski space, as shown below. For events with timelike separation, communication between S and M is possible a sub-light velocity. The two events can be linked by sub-lightspeed signals.

2. *Lightlike Separation*: if $s^2 = 0$ then $x/t = c$ and $x'/t' = c$. Events with lightlike separation lie along the lightline and communication between them by light pulses is possible, but no slower-than-light communication is possible.

3. *Spacelike Separation*: if $s^2 < 0$ then $x/t > c$ and $x'/t' > c$. Events with spacelike separation lie below the Minkowski space light ray and communication between them by light pulses or any other means is not possible. Neither event can be linked to the other.

Figure 9
Spacetime Classifications



6. Force, Mass, Momentum and Energy

The most famous equation in physics is $E = mc^2$: Energy equals mass times the speed of light squared. This is a direct outgrowth of Special Relativity. We have two important pieces of information necessary to derive that famous equation: All motion in spacetime is at the speed of light, and the Law of Conservation of Momentum applies even when there is relative motion.

6.1 Relativistic Mass

Most particles have mass; photons and neutrinos are the most well known exceptions. Rest mass, denoted by m', is the amount of "stuff" in an object, independent of the effect of gravity (which determines the weight of that stuff). Mass is defined by the force required to accelerate an object: using Newton's second law of motion $F = ma$ (force equals mass times acceleration), rest mass is $m' = F/a$. The SI units are kilogram for mass, meter per second per second for acceleration, and the Joule for force or energy. [Note: Force is defined as the derivative of momentum ($p = m'v$) with respect to proper time, i.e. $F = d(m'v)/dt'$.]

As the velocity—hence momentum—of a particle increases the force required to maintain a specific rate of acceleration increases. We have seen that *relativistic mass* is $m = \gamma m'$, where m' is the proper mass (rest mass) and γ is the time dilation factor, i.e. $\gamma = 1/\sqrt{1 - \beta^2}$). Note that as an object's velocity approaches the speed of light, its mass becomes infinite.

While $m = \gamma m'$ is an exact relationship, there is a convenient approximation, that helps to understand the components of relativistic mass. Because γ is (approximately) equal to $1 + \frac{1}{2}\beta^2$ the relationship in Box 7 can be derived.

Box 7
Relativistic Energy-Mass Equivalence

$$E = m'c^2 \quad + \quad \tfrac{1}{2}\beta^2 m'c^2$$

↑ Rest Energy ↑ Kinectic Energy

The assumption underlying the common description of energy, $E = m'c^2$, is that the object is not in relative motion, i.e., $\beta = 0$. When there is relative motion at low velocities, the approximation in Box 7 is valid. At high velocities, the exact equation for relativistic mass, $E = \gamma m'c^2$, should be used

6.2 Momentum and its Conservation

In classical mechanics the *momentum* of an object is an important concept. Momentum (p) is rest mass times velocity: $p = m'v$. Momentum, like energy-mass, is conserved, that is, it is not changed by any fragmentation of an object or by integration of pieces.

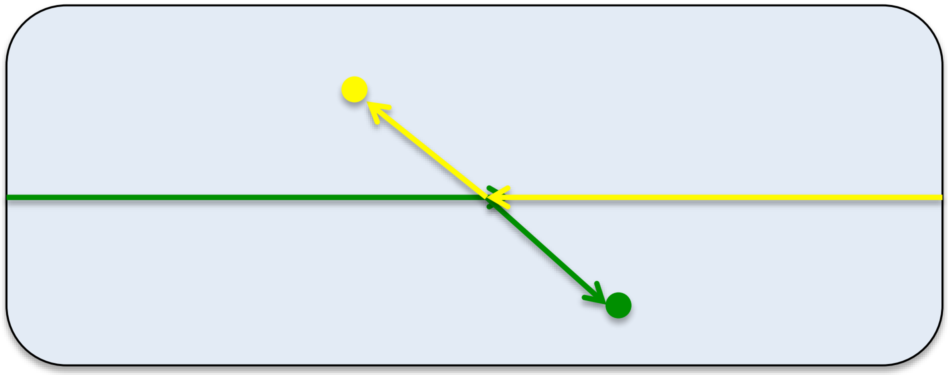
Suppose that a system has an initial state with objects moving about, each with a momentum $p_i = m_i v_i$, giving a total net momentum of $P = \Sigma m_i v_i$. Suppose also that the rest mass of this system is $M = \Sigma m'_i$. For example, the planets, asteroids, rocks, and dust in our solar system are all objects, each with its own mass and momentum relative to Earth. The Law of Conservation of Momentum (LCM) states that any collisions between objects, though they might change the mass and velocity of individual objects, will not change P, the total momentum. Ignoring conversions of mass into energy (as into heat generated by collisions) those collisions will also not change the system's total mass, M.

The LCM is very well established in classical mechanics, but it was not known whether it held up under relative motion, when different inertial frames exist "simultaneously." Einstein postulated that the LCM applies even then, and from that derived the notion that mass increases with velocity. Experiments have confirmed Einstein's conjecture.

A simple thought experiment is constructive. Consider a billiard table on which two balls, each with the same mass and equal speed, are traveling directly at each other. The one going left to right (ball G, green) has velocity v, and the one going right to left (ball Y, yellow) has velocity –v. The system (billiard table) has a zero net momentum: $P = mv + m(-v) = 0$. After the two balls collide—that point is called *the center of momentum*—they reverse their directions and have lower but still equal speeds, v* but opposite directions: the velocities are -v* (left ball) and +v* (right ball). The net momentum $P = m(-v^*) + mv^*$, is still zero. Momentum is conserved.

Now consider a similar situation in which the two balls are coming at each other, but slightly off center so that now they will glance off at lower speed but exactly opposite directions, as in the layout below.

After the collision, G and Y both go in opposite direction at velocities v* and –v*, where $v^* < v$ because of the loss of energy due to the release of sound and heat in the collision. The new net momentum is $P^* = mv^* + m(-v^*) = 0$. Momentum of each ball has changed because of the change in direction and speed, but total momentum is unchanged.



This is the essence of the Law of Conservation of Momentum in a classical framework. But what are the implications if the Law of Conservation of Momentum is valid in a relativistic framework, when the velocities of Y and G are near light speed? To see this we repeat the billiard table below.

6.3 Relativistic Force, Mass, and Acceleration

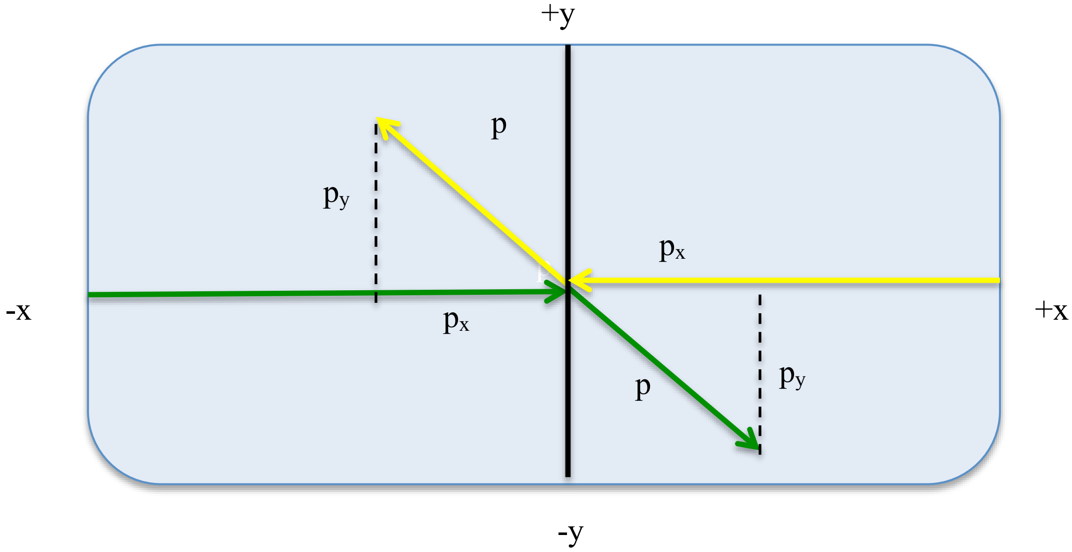
The billiard table thought experiment, when translated to high velocities, is a simplified picture of two protons colliding in the Large Hadron Collider: the protons approach each other along the x-axis, collide, glance off at the same angles, and proceed at lower but equal speeds in a new direction. We assume that there is no decay into smaller particles (as would occur in a proton accelerator) and that no energy is lost in the collision.

The motion of each proton after the collision is a combination of motions along both the x-axis and the z-axis. This is shown below, where we reproduce the billiard table and explicitly note the vertical y-axis and the horizontal x-axis that intersect at the COM.

In Figure 10 the post-collision arrows are momentum vectors in the two dimensions, so p (the length of the arrow) is momentum, p_x is momentum along the x-axis ($m'v_x$) and p_y is the momentum along the y axis (v_y). The *proper* net momentum of, say, the green proton, is defined as $p' = m'v = m'\sqrt{(v_x^2 + v_y^2)}$ where m' is the rest mass of the proton.

What is the momentum as measured by a stationary observer, say a physicist sitting at the center of momentum (COM)? Well, velocity is an invariant attribute so he and the proton would both measure the same velocity of the proton. But the mass is not invariant: the COM will measure the mass as γm'; this is the relativistic mass. So the COM measures the momentum as $p = \gamma m'v$.

Figure 10
Proton Collision in a Collider



Why does relativistic mass dilate by the same proportion as does time or distance? The answer is that relativistic mass is different from rest mass precisely because of the dilation of time. To see this, recall Newton's second law: F = m'a. *Proper force*, the force seen by the proton, is the derivative of its momentum with respect to proper time, i.e. F' = d(m'v)/dt' = m'a where a = (dv/dt') is proper acceleration. In order to convert this to COM's measure of the acceleration (i.e., dv/dt) we only need to note that dt'/dt = γ, then multiply (dv/dt') by γ to get COM-force: F = γm'a. Since mass is defined as F/a, we have "discovered" that for a stationary observer, the mass of an object in motion is γm'. [The same result is found when the decomposition into x and y velocities is considered: the only change is that acceleration is defined as $a = (v_x a_x + v_y a_y)/v$; acceleration is a weighted average of the x-acceleration and y-acceleration, with the velocities as weights.]

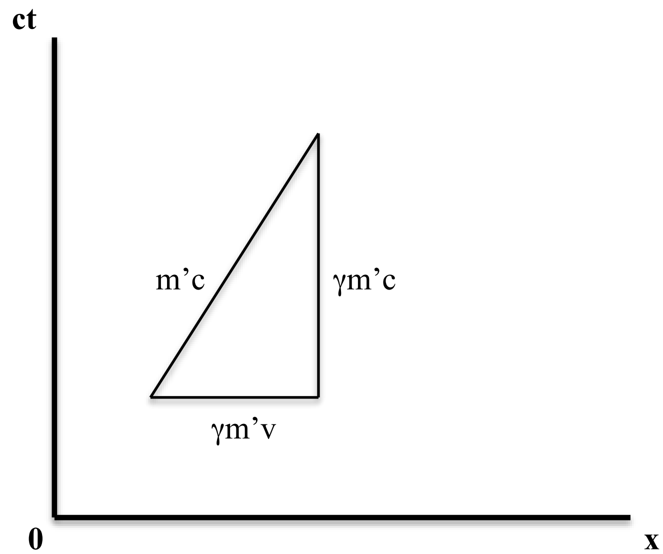
Note that it is not really the mass that is increasing. The proper mass is unchanged; it is the time unit of acceleration that is changing due to translation from moving to stationary observers. But whatever the source, the effect is *as if* mass had increased—it takes greater force to accelerate a moving object—and it is conventional to attribute this to mass.

6.4 The Equivalence of Mass and Energy: Why is $E = mc^2$?

During 1905, Einstein's *annus mirabilis*, he published four world-changing papers. Included were his paper on Special Relativity and a short two-page follow-up paper on the equivalence between mass and energy in which he concluded that $E = mc^2$.

We have seen though the breakdown between space and time movements depends on the observer, all objects move through spacetime at the rate c, the speed of light. Thus, in spacetime, a moving observer's proper momentum is p' = m'c and that momentum, as seen by a stationary observer traveling through time only, is p = γm'c. This allows us to draw Figure 8 in which the length of the hypotenuse is M's momentum (m'c), the vertical leg is momentum through time (γm'v)—this is S's view of M's momentum. The horizontal leg is momentum attributable to motion through space (γm'v).

Figure 11
Momentum in Spacetime



If we multiply each leg by c—an invariant number—we get Figure 12.

Figure 12
Energy in Spacetime

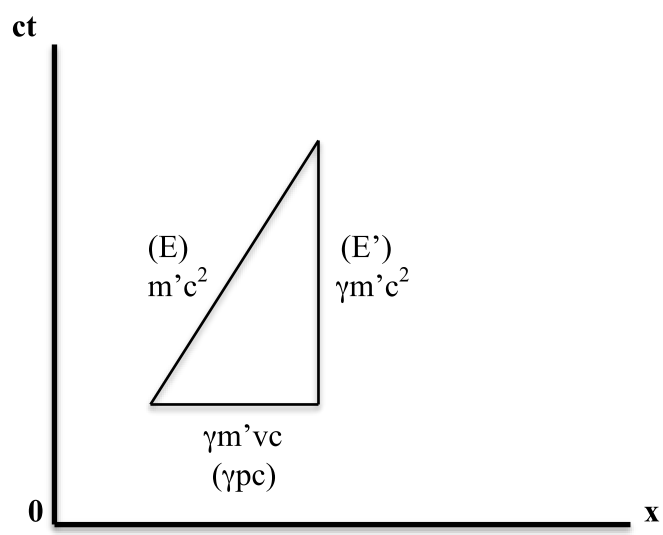


Figure 12 converts Figure 11's momentum into a view of energy. The hypotenuse in Figure 12 is the *rest energy* or *proper energy* as seen by M. The vertical leg (travel through time only) is *relativistic energy*, composed of both rest energy and kinetic energy, as measured by S. It implies energy-mass equivalence as shown in Box 8.

Box 8
Energy-Mass Equivalence

$$E = \sqrt{(E'^2 + \gamma p'c)}$$

where

$E' = m'c^2$ is rest energy (in Joules)
$p' = m'v$ is proper momentum
$E = m'c^2$ is relativistic energy (in Joules)
$c = 3{,}000{,}000$ m/sec
m' = rest mass in kilograms
v = velocity in m/sec

- One kilogram of mass is equivalent to 9×10^{16} Joules (= 1kg x $300{,}000{,}000^2$ m^2/sec^2) of rest energy. This translates to 25 billion kilowatt-hours of power, or 0.61 percent of U.S electricity consumption in 2010. 100 kilograms of matter, if converted with perfect efficiency into electricity, would provide 61 percent of U.S. electricity!
- A 1000 kilogram missile has been launched by the Klingons at the U.S.S. Enterprise. Its rest energy is 9×10^{19} J. Its velocity is 99 percent of light speed, giving $\gamma p'c = 21.06\times10^{19}$ J. The percentage increase in energy due to motion is (approximately) a miniscule 1.17×10^{-17} percent, or 1,053 Joules. This is 0.003 KWH of kinectic energy compared to 2.57×10^{14} KWH of rest energy. It takes that last one percent of light speed (from β of 0.99 to 1.00) to really make a difference to the impact of the missile on the Enterprises blast shield

Relativistic and rest energy are equal when momentum is zero; the difference between energy and rest energy increases as velocity—and, therefore, momentum and the Lorentz factor—increase. The kinetic energy of the object is the contribution of γpc to E.

7. Special Relativity Arising From Earth's Rotation

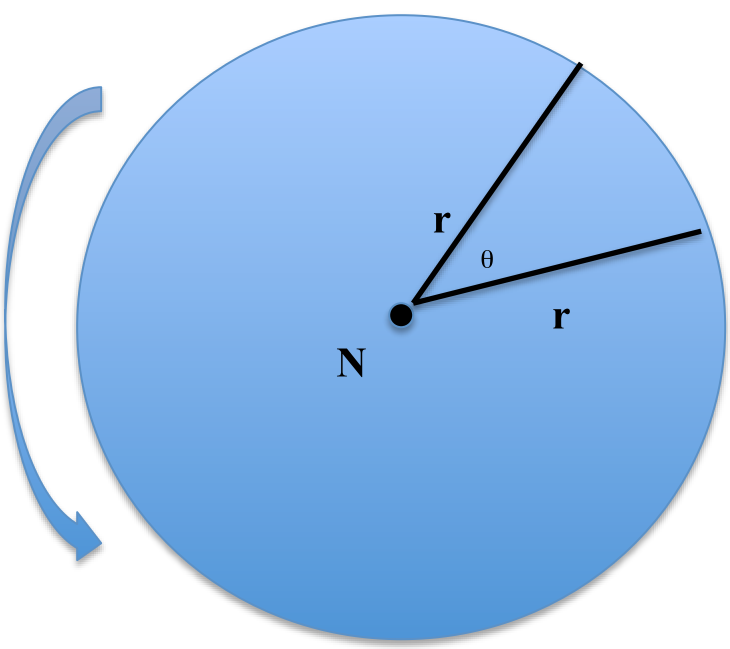
In his 1905 paper on Special Relativity Einstein suggested a test of his theory: "…thence we conclude that a spring-clock at the equator must go more slowly, by a very small amount, than a precisely similar clock situated at one of the poles under otherwise identical conditions." If true, this would create a number of practical problems, among them the need to make relativistic adjustments to synchronize clocks at different latitudes.

It turns out that for reasons beyond Einstein's understanding at the time, this is not true: clocks at different latitudes do nut run at different paces. but this conjecture is an opportunity to demonstrate the effects of relativity under conditions of constant rotation rather than of constant directional motion.

7.1 Relativity Effects on a Rotating Disk

Suppose that a perfectly round disk is rotating at a constant speed, as in Figure 13. To make it concrete, imagine that this disk is the cross-section of the Earth at the equator. Of course, the earth is not a perfect circle at the equator, and when seen in three dimensions it is not a perfect sphere—it is oblate, or bulged at the equator. But imagination can make these imperfections disappear.

Figure 13
The Earth's Equatorial Disk



The Earth has an equatorial radius of 4000 miles (6, 400km, 64,000,000m), a circumference of roughly 25,000 miles (40,000 km, 40,000,000 m), and a rotation period of 23.9 hours (86,200 s). Its average speed of rotation is 1,050 miles per hour (1,675km per hour, 4,65 meters per second).

In Figure 13 N is the North Pole and the disk is the equatorial Great Circle as seen from the pole. The Earth rotates counterclockwise from N (eastward). The figure shows two radius lines as well as the angle between them (θ). That angle increases as the Earth rotates; the speed of rotation is the *angular velocity*, denoted by ω ($\omega = d\theta/dt$). The linear velocity, denoted by v_L ($v_L = r\omega$), is the velocity of rotation in meters per second; it is this velocity that is used to define rapidity as $\beta = r\omega/c$, so the modified dilation factor is

Box 9
Dilation On a Rotating Disk

$$\gamma = 1/\sqrt{[1 - (r\omega/c)^2]}$$

r = radius of Disk (meters)
ω = angular velocity (radians per second)
rω = linear velocity

The essential data to compute the dilation factor are $r = 64 \times 10^6$ meters, $c = 3 \times 10^8$ meters per second, and ω = 7.29x10-6 radians per second. Thus $\beta = 1.552 \times 10^{-9}$. This is an extremely low rapidity, so we would expects a tiny dilation factor. And that is just what we get: $\gamma = 1.000000000121$. One rotation of the Earth over 86,200 seconds would create a time lag at the pole of 1,043 nanoseconds in coordinate time.

Thus Einstein's conjecture is supported by Special Relativity. The time difference would be slower at higher latitudes because the radius of the disk is smaller, reducing the linear velocity, but SR predicts that clocks will run slower at the poles and faster at the equator.

7.2 The Hafele – Keating Experiment

In 1971 an experiment was done to test time dilation. Four atomic clocks were put on an airplane and flown around the world. At the end, the time on each clock was compared to the time on a companion clock that was fixed in place at the Naval Observatory in Washington D.C. This experiment does not address the possibility that time runs faster at lower latitudes; rather, it tests the prediction of time dilation when relative velocities differ.

The test was done twice, first with the plane going eastward with the Earth's rotation, then again with the plane going westward against the Earth's rotation. The reason is that due to the Earth's rotation the westward velocity of the plane was higher relative to the Observatory than the eastward velocity.

The HK experiment considered the effects of General Relativity as well as Special Relativity. It turns out that these are partially offsetting. General Relativity predicts that a clock runs slower faster the weaker its gravitational field (hence the higher its altitude), while Special Relativity predicts that a moving clock will run faster. The predicted time gains (in nanoseconds) on the air borne clocks were -40 ± 23 nsecs eastward and +275 ± 1 nsecs for the westward flight (the ± is a reliability range). The observed gains were -59 ±10 and +273 ± 7, respectively.

The HK experiment was hailed as a definitive test of Special Relativity. However, as will always happen when seminal experimental results are announced, there were some legitimate criticisms associated with the accuracy of the atomic clocks, the difficulty of piecing together results from different legs of the trip, and adjustments made by HK to the raw data. This experiment has been replicated by the BBC and NPL (2005), and again by NPL/BBC (2011) [NPL is Britain's National Physical Laboratory]. Each time the HK results have been supported with increased predictive accuracy.

The HK experiment was hailed as resounding proof of Special Relativity's time dilation prediction. There is still a cottage industry of critics, a few with credentials (see the paper by Wang listed in the references) but their influence has waned and they are now relegated to the "crank" category.

7.3 Does Latitude Matter?

We have seen that Special Relativity predicts clock speed differentials at different latitudes. But in fact, no such differentials are observed. The reason is that Special Relativity is not the whole story: it applies when masses are insignificant, as when two clocks are free-floating in space at different velocities. But General Relativity becomes important when mass-induced gravitational effects exist. The Earth is very massive, and General Relativity dominates the effects of Special Relativity. It turns out that when the effects of both Special and General relativity are considered, the two effects precisely cancel out and there are no on latitudinal time differences.

As noted above, the fact that latitude does not matter to clock speeds does not negate the Hafele-Keating results. Those applied not to latitudinal differences but to differences in relative velocities of objects at different altitudes but at the same latitude. So while we don't need to consider relativity in synchronizing clocks at different latitudes, we do need to consider relativity for objects orbiting the Earth, such as GPS satellites.

8. Summary

In 1905 Einstein published four papers that revolutionized our understanding of the way nature works: *The Electrodynamics of Moving Bodies* outlined what has become known as the Theory of Special Relativity; *Does the Inertia of a Body Depend Upon its Energy Content?* extrapolated from the first to report that mass and energy are equivalent, and that the transformation between them allows the conversion of small pieces of matter into very large amounts of energy: one kilogram of matter can generate 25 *billion* kilowatts of electrical power if the conversion of mass to energy were perfectly efficient.

Have you ever watched a plane take off and seen that as it accelerates it becomes shorter, heavier, and the passengers age at a slower rate? Well, neither have I. It is happening, but the speeds are too low for us to notice the minute relativistic effects. However, if we were watching the U.S.S. Enterprise, Star Fleet Command's most famous vessel, we would certainly notice those effects as it boosted into "warp drive." Special Relativity is real!

Its reality has been demonstrated in over 100 years of experiments. Clocks on airplanes do tick more slowly relative to clocks on the ground, and, as predicted, the degree of time difference is greater for westward-bound planes than for eastward-bound planes; the mass of particles in Colliders does increase as their velocities—and energies—get accelerated to near light speed; distances traveled by decaying particles, like muons, do increase as their velocities rise, as does their half-life. Nothing has ever been observed to move at greater than the speed of light. Special Relativity has been tested often, and has never failed. Even recent reports that new experiments revealed particles traveling faster than light, thereby disproving relativity, were found wrong for a very interesting reason: *the authors forgot to adjust for relativistic effects!*

Even so, hope springs eternal! Books are still written on the Relativity hoax, and the internet is rife with websites devoted to debunking relativity; and "paradoxes" still are found that are reputed to undermine the theory. But these are the imaginings of pseudo-scientists who don't understand the deep complexities in Special Relativity. But Relativity is rife with philosophical and scientific conundrums, and those paradoxes and "inconsistencies" have all been resolved when they are subjected to careful scientific scrutiny.

We are all influenced by relativity, usually in ways we would never see. Global Positioning Systems rely on relativistic adjustments to coordinate times and distance measurements between satellites and ground stations: without those adjustments, cumulative navigation errors would create havoc. Scientists make relativistic adjustments that make experimental results make sense.

Do we all have to understand the Theory of Special Relativity? The answer is a resounding "No!" It applies only in situations of constant velocity, and its effects are only on scales far different from those we normally experience: velocities near light speed. Physicists, electrical engineers, and their ilk certainly need to understand relativity—but you and I don't.

Still, *someone* has to understand those effects, because we rely on them in our technology, and, therefore increasingly, in our lives. And it behooves us to understand what our technology is doing. So I hope that someone finds this review of Special Relativity useful in sorting through the clutter and understanding one of the most important scientific discoveries of the 20th century—and perhaps of all time.

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Note: the Cox-Foreshaw and the Greene sources are extremely readable and are highly recommended

Appendix
Units of Measurement and Conversions

	SI	Poundal	Conversions
Mass	Kilogram (kg)	Pound (lb)	1 kg = 2.2 lbs
Distance	Meter (m) Kilometer (km)	Foot (ft) Mile (M)	1 m = 3.28 ft 1 km = 0.62 M
Time	Second (s)	Second (s)	---
Force	Newton (N) Dyne (D)	Pound-Force (lb_F)	1 N = 1 kg m/s^2 = 1x10^5 Dynes = .225 lb_F
Energy (Work)	Joule (J) Dyne (D)	Watt (W) BTU	1 J = 1 N m = 1 W/s 3.5x10^5 J = 1 KWH 1 KWH = 3,412 BTU
Light Speed	299,792, 458 m/s 299,792.458 km/s	983,571,056 ft/s 186,282 M/s	--- ---