Rock and Roll:

The Lateral Stability of Vessels

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Contents

Introduction[1]

This analysis focuses on the basic physics of lateral stability. We ask several questions: Will a heeled-over boat return to the vertical? If so, how long will it take? What is the influence of wave shape and frequency on the answer? What can be done to improve lateral stability? There are other dimensions of stability that are not considered. For example: pitching and yawing are not considered; swamping due to low freeboard is ignored; the effect of irregular waves (steep faces, varying frequencies or direction) is left out.

Summary

- **Stability is the result of two offsetting forces: gravity (downward) and buoyancy (upward). The result is a twisting force (torque) that restores stability; the torque increases with the heeling (roll) angle.**
- ***Rotational acceleration*, measuring the rapidity of restoration to neutrality, is proportional to the *righting arm* and inversely proportional to the square of the beam. Beamy boats, other things equal, are "tender," returning to neutrality more slowly than "stiff" boats.**
- **The "tenderness" of a vessel is measured by its *roll* period, which is directly proportional to the square of the beam, and inversely proportional to the square root of the *metacentric height* (and, therefore to the boat's righting arm).**
- **Roll period is independent of heeling angle: a slightly heeled vessel and a very heeled vessel will return to vertical in the same period, other things equal.**
- **Waves on the beam induce roll angles that come from two sources: the steepness of the arriving wave (wave shape), and the rotational acceleration (roll period): steep wave fronts and short roll periods induce greater roll angles**
- **The roll angle induced by waves is at a maximum when waves arrive at the *resonant frequency*: higher or lower wave frequencies induce smaller roll angles than the resonant frequency.**
- **The maximum roll angle is directly proportional to wave height inversely proportional to the coefficient of friction between boat and water, and inversely related to the boat's mass.**

[1] The equations reported in these notes cannot be used without identifying the units of measurement. They are: meters for distance, seconds for time; kilograms for mass; radians for angular movements (rotations).

None of these results will be new insights to the seasoned mariner. The objective here is not to state that they exist but to explain *why* these well-known effects occur.

Fundamentals of Stability

Figure (1a) shows an unheeled vessel: the *Center of Buoyancy* (B), the *Center of Gravity (G)* and the *Metacenter* (M) all vertically above the Keel (K). This vertical line is called the *metacentric line*. Bouyancy exerts a vertical upward force (F_b) from point B; gravity exerts a downward force (F_g) from point G. The two forces are equal and offsetting ($F_b = -F_g$) so the vessel sits in the water with the weight of water displaced (its displacement) equal to the vessel's weight.

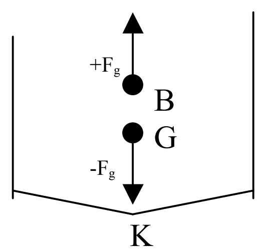


Figure I
Unheeled Vessel

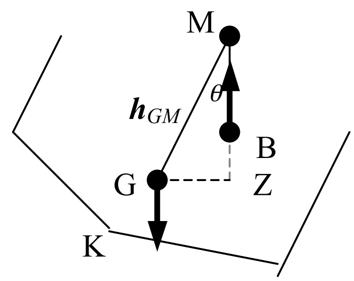


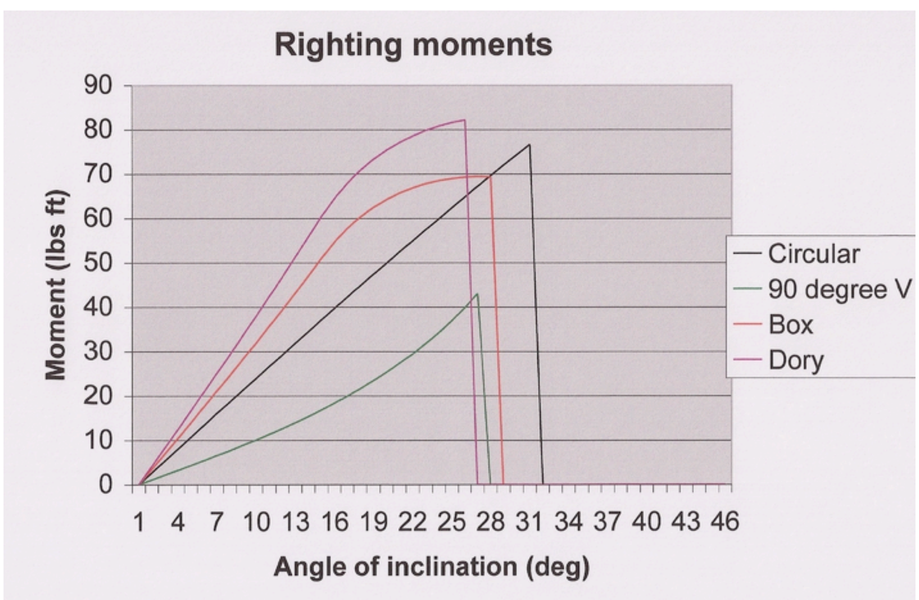
Figure II
Vessel Heeled by θ°
Righting Arm (GZ) = $h_{GM}\sin(\theta)$

Consider the unheeled vessel in Figure I. The central axis of the boat is BGK, where K is the keel, G is the center of gravity—the axis of rotation around which the vessel rotates when it heels, and B is the center of buoyancy. The downward force of gravity at G is $-F_g$; it is exactly balanced by the upward force of buoyancy (F_g) at B so the boat sits erect on the water at a keel depth that displaces a weight of water exactly equal to the weight of the boat.

Now turn to Figure II. The boat is heeled to starboard from the central axis at angle θ. The vertical forces of gravity and buoyancy are unchanged. The center of gravity is unchanged but the center of buoyancy has shifted to starboard and something important has happened: because B is now to the right of G a counterclockwise force (torque) is created—buoyancy lifts the starboard side while gravity pulls down the at its center, twisting the boat counterclockwise and countering the roll. The boat is stable, tending to return to the vertical.

In Figure II the point M is introduced. This is called the *metacenter* and it is always directly above the center of buoyancy.[2] The *metacentric height* (denoted GM, or $\boldsymbol{h}_{GM}$), is the distance along the central axis between the metacenter M and the center of gravity G. In Figure II the point Z is directly below the metacenter at the height of the center of gravity, creating a right triangle GZM. The leg GZ is called the *righting arm* and the torque created by the roll is proportional to the its length: specifically, the torque is proportional to $\mathbf{h}_{GM}\sin(\theta)$. The righting arm increases from zero at θ=0° to $\mathbf{h}_{GM}= 1$ at θ=90° (where the righting arm length is maximized). As the roll angle increases the righting arm lengthens and the opposing torque increases. A stable vessel always creates enough torque to right itself, no matter how extreme the roll angle.

The righting arm depends on the hull shape and the heeling angle. The chart below shows the righting arm length for several hull shapes, as a function of the heeling angle. A box-shaped hull reaches the highest moment arm, but capsizes at the lowest heeling angle because the center of gravity and the center of buoyancy are close. A circular (rounded) hull capsizes at the highest heeling angle, but has a low righting moment at stable heeling angles, thus demonstrating the well-known fact that round-bottomed boats wallow but are more stable.



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[2] The metacenter is calculated as KB + (I*/V), where KB is the distance from the keel to the center of buoyancy in figure (1a), V is the volume of water displaced (in cubic meters), and I* is the *second moment of area* (a measure of resistance to torsion) of the displaced water.

Torque and Stability

Suppose a vessel is heeled because of wind; waves will be considered later. In the figure below we show an enlargement of Figure II—a vessel heeled to starboard by angle θ. The force applied to the center of gravity, G, is the downward force $F_g = -mg$ (m is the vessel's mass and g is the gravitational constant). The center of buoyancy exerts an upward force of +mg, so the sum of the two forces is zero.

Note the point Z directly right of the center of gravity and directly below the center of buoyancy. The horizontal distance GZ is called the *righting arm*; the length of the righting arm is $\mathbf{r} = \mathbf{h}_{GM}\sin(\theta)$. The righting arm acts as a lever pivoting on the center of gravity, and the force of buoyance acts as a force pushing the point Z upward. The result is a twisting force—torque—operating on the righting arm to pivot it counterclockwise. The force of buoyancy is $F_T = mg$. The amount of torque, denoted τ_1, is defined as $\tau_1 = \mathbf{r}F_{T,}$ where $F_T = mg$ is the force of torque. Thus, $\tau_1 = mg \bullet h_{GM}\sin(\theta)$.

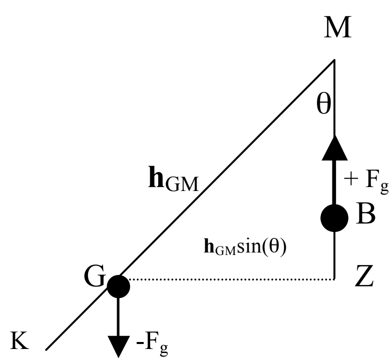

Figure III

There is a second definition of torque (which is why the first is denoted τ_1). Torque is the moment of inertia (denoted I) times the angular acceleration (denoted α), so $\tau_2 = I\alpha$. Because the two torques must be equal, we have the following equation:

$$\text{(2a)} \qquad I\alpha = mg\mathbf{h}_{GM}\sin(\theta)$$

Because, by definition, $\alpha = d^2\theta/d^2t$, (2a) is an equation of motion relating the acceleration of θ to the level of θ, assuming all other influences are independent of θ. The moment of inertia is defined as $(m\bullet R_g^{\ 2})$, where R_g is the vessel's *radius of gyration*. The radius of gyration depends on the specific characteristics of the vessel (hull shape, keel length, density structure of equipment and other internal items, and so on): for a half shell hull it is simply ½R (R is the outside radius or beam); for a half-cylinder it is .71R. Thus, the equation of motion for the roll angle is

$$\text{(2b)} \qquad \alpha = [g\bullet h_{GM}\sin(\theta)]/R_g^{\ 2}$$

For example, consider a boat with an open shell hull. The radius of gyration is ½beam and $\alpha = -4(g/\text{beam}^2)\sin(\theta)$. For any given heeling angle, doubling the beam decreases the angular acceleration by 75 percent, i.e. the rate at which the boat's roll angle increases drops by 25 percent. The angular acceleration is positive, meaning that the roll angle increases at a faster rate as the vessel rights itself.[3]

As noted above, for small values of the heeling angle, the metacenter can be treated as fixed. But at sufficiently high heeling angles the metacenter shifts down the metacentric line. This fact is essential to vessel capsizing. The critical heeling angle is that for which the metacenter is at the center of gravity: the metacentric height and the righting arm vanish.

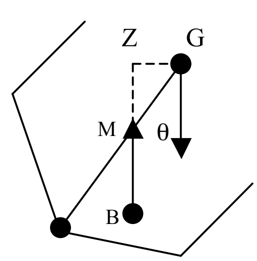

Figure IV   Capsizing Vessel

[3] Rotational velocity, denoted by ω, is defined as dθ/dt. Rotational acceleration is $\alpha = d\omega/dt = d^2\theta/dt^2$. When we refer to g in the text we mean the positive value: g = 9.8 meter/second^2 or 32.2 feet/ second^2.

A boat in this situation will neither right nor capsize. Higher heeling angles create a negative metacentric height and negative righting arm, as in Figure IV, showing a capsizing vessel. The center of buoyancy is below the center of gravity so the metacenter is below the center of gravity and the righting arm GZ is negative. The positive force of buoyancy now reinforces the negative force of gravity, the boat continues to roll, the heeling angle increases further, and the boat capsizes.

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Roll Period

We have investigated the speed at which a roll angle is corrected. Now we look at the time it takes for a stable heeled vessel to return to neutrality—the *roll period*, defined as the theoretical time for a vessel with a given hull radius to rotate through a complete cycle from (say) vertical to θ° starboard, back to port, then back to vertical. We will see that roll period does not depend on the initial heeling angle: surprisingly, the period from high roll angles is the same as that from low angles: high roll angles just have a correspondingly faster angular velocity.

Roll time is defined as $T = 2\pi/\omega$ where 2π is the radians in a complete cycle, where ω is the angular velocity (radians per second); this is the standard definition of the time required for any oscillation to complete one cycle. It turns out that if we solve the equation of motion (equation 2b) the angular velocity is $\omega = (1/R_g)\sqrt{(g \cdot \mathbf{h}_{GM})}$ so

$$\text{(3a)} \qquad T = 2\pi R_g/(g \cdot \mathbf{h}_{GM})^{1/2}$$

Thus, the roll period is directly proportional to the radius of gyration—longer for beamier boats—and inversely proportional to the square root of the metacentric height—shorter for more stable boats.[4] *Note that only the metacentric height and the radius of gyration affect roll time; neither the roll angle or the boat's mass matter.*

For a solid half-cylinder the radius of gyration is $R/\sqrt{2}$, where R is the hull's radius. So the roll period is

$$\text{(3b))} \qquad T = \pi R/(g \cdot \mathbf{h}_{GM}/2)^{1/2}$$

This assumes a solid half-cylinder hull. In general, the radius of gyration depends on hull design and ship size; an approximation is $R_g = kR$, with $k = 0.4 < k < 0.55$, depending on ship size and hull design. For a semicircular shell k = 0.5; a common value for large vessels (warships) is k = 0.40; and, as we have seen, k = .71 for a solid half-

[4] This uses $\sin(\theta) = \theta$ as a linear approximation appropriate for "small" values of θ.

cylinder hull. For example, consider a boat with k = .45, a 7 meter beam, and a 2 meter metacentric height. The roll time is approximately 16 seconds. This would be a “tender” boat.

Wave Frequency and Resonance

Thus far we've assumed that our vessel has been in calm water, heeling only because a gust of wind hits it; that is, the vessel is in still water, its neutral position is vertical, and a single impulse has created the heel. A more realistic scenario is that continuous waves have created the roll.

Introducing waves raises an essential feature of any waves, whether water waves or electromagnetic waves—the phenomenon of *reasonance*. As waves strike the beam at increasing frequency (going from, say, long ground swells to higher frequency) the amplitude of the roll angle will increase. The roll angle will reach a maximum when wave frequency—defined as wave length divided by wave period (time between wave crests)—is at the *resonant frequency*,

To set the stage, let λ is the wave length (distance in meters between crests) and ω' be the wave angular velocity (the rate per second at which a wave proceeds from crest to crest). This is akin, but not equivalent, to the angular velocity found in the discussion of roll period. Then $T' = 2\pi/\omega'$ is the wave period (in seconds), $f = 1/T'$ is the wave frequency (seconds per wave), and $\upsilon = \lambda/T'$ is the wave velocity (meters per second). The angular velocity is $\omega' = 2\pi\upsilon/\lambda$, where υ is the wave velocity (in meters per second); in general $\upsilon = \lambda/T'$ but a deep-water approximation is $\upsilon = \lambda\sqrt{g}/2\pi$. Thus, in deep water a wave with 3-meter length travels at 2.16 meters per second (7.8 kilometers per hour).

Consider a chain of waves coming on the beam in a regular sinusoidal pattern and creating the following equation for the height of the wave (φ):

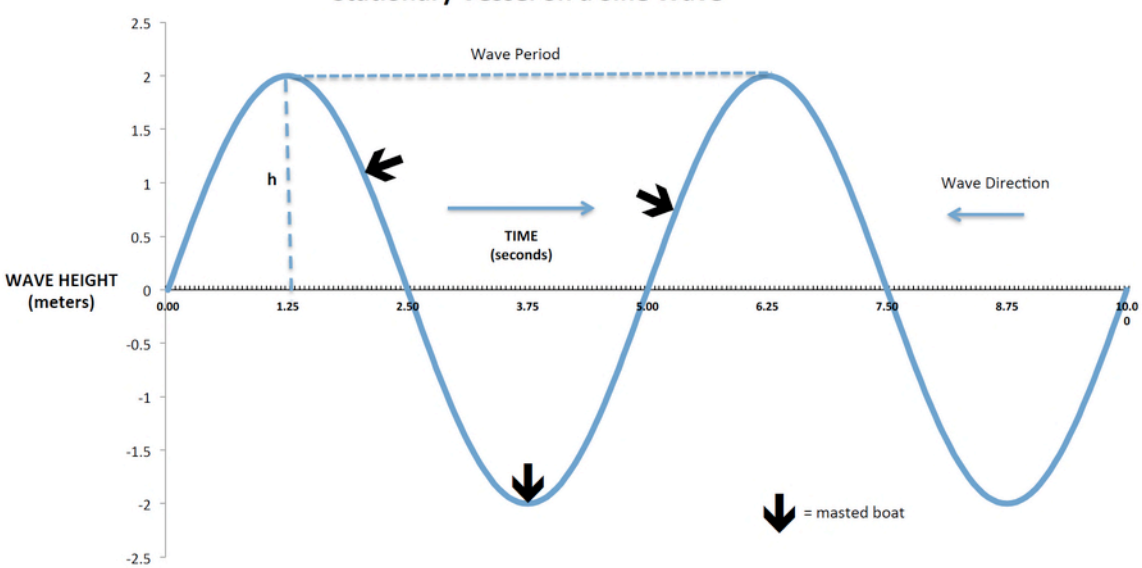
$$\text{(4a)} \qquad \varphi(t) = \mathbf{h}\sin(\omega' t)$$

where **h** is the wave's amplitude in meters (this is not the h of previous sections). Thus, the wave height (trough to crest) is 2**h**. The wave pattern is sinusoidal, as in Figure V, and the wave slope at each moment is ($d\varphi/dt$) is $\omega'\mathbf{h}\cos(\omega' t)$.

In Figure V there are two full sinusoidal waves with five-second periods and amplitude **h**; waves are moving from right to left. The symbol 🡇 represents a masted boat in a vertical position. The boat is stationary and we see that it is always perpendicular to the wave slope: rising and rolling leftward on a rising wave and falling and rotating rightward on a falling wave. At about two seconds after a wave enters on the right, the boat (shown on the left) is heeled to the right on the backside of a wave; at 3¾ seconds it is in the trough (the center figure); at about 5½ seconds it is on the front face of the next wave. The perpendicularity to wave face holds at each moment *if* the boat instantaneously adjusts to the changing wave face slope, but in reality the boat will always be recovering from its prior roll angles. Thus, the analysis of a boat's roll depends on its position on the wave and on its roll period

This representation of roll due to waves is highly stylized. It ignores the reality of non-sine waves as well as the influence of breaking waves, waves arriving from different directions, and of internal adjustments like shifting payloads. Even so, it is a useful approach to the fundamental issues raised by wave action.

Figure V
Stationary Vessel on a Sine Wave



Denny (2008)[5] shows that (with some simplifying assumptions) the roll angle from both sine waves and heeling is

[5] Denny's analysis (pp. 161-163, notes 5, 6 on p. 238) is based on a number of simplifying assumptions, among them: The hull is assumed to be a solid half-cylinder and wave velocity is $v = \sqrt{g\lambda/2\pi}$ (the deep water value).

$$\text{(4b)} \qquad \theta(t) = \mathbf{a}\cos(\phi + \omega' t)$$

in which **a** is the amplitude of the roll angle and ϕ is the phase angle, i.e., the time delay between the wave face change and the boat's roll angle change; $\phi = 0$ will generate the perpendicularity sown in Figure V.

The roll angle's amplitude is a function of wave and boat characteristics that are embedded in **a**. Denny shows that the equation of motion for the roll angle, θ, is a second order differential equation[6] and that the amplitude of the roll angle is

$$\text{(4c)} \qquad \mathbf{a} = \Omega_1^2/[(\Omega_0^2 - \omega'^2)^2 + \mathbf{b}^2\omega'^2]^{1/2}$$

where $\omega' = \sqrt{[2\pi g/\lambda]}$, $\Omega_0^2 = gh_{CG}/R_g^2$, and $\Omega_1^2 = \sqrt{[(2\pi\mathbf{a}/\lambda)(gh_{CB}/R_g^2)]}$.

The parameters h_{CG} and h_{CB} are distances from the center of the deck (the flat side of a solid half-cylinder) to the centers of gravity and buoyancy, respectively; **b** is the coefficient of friction between the hull and water.

Knowing the parameters allows calculation of the roll angle as waves arrive and depart. The roll angle will be at a maximum when waves arrive at the resonant frequency, that is, when $d\mathbf{a}/d\lambda = 0$. From equation (4c) we find that this occurs at the angular velocity shown in (4d), the wave length shown in (4e), the roll period in (4f) and the wave frequency in (4g):

$$\text{(4d)} \quad \omega' = \sqrt{[gh_{CG}/R_g^2]}$$

$$\text{(4e)} \quad \lambda = 2\pi R_g^2/h_{CG}$$

$$\text{(4f)} \quad T' = \lambda/\upsilon$$

$$\text{(4g)} \quad f = \upsilon/\lambda$$

The maximum roll angle, $\mathbf{a}_{max}$, is when the amplitude is maximized, or

$$\text{(4h)} \quad \theta_{max} = \sqrt{[gh_{CG}h_{CB}h/bR_g^3]}$$

[6] The equation is $d^2\theta/dt^2 + \mathbf{b}(d\theta/dt) + \Omega_0^2 = \Omega_1^2 \bullet \cos(\omega' t)$. This is the equation for forced damped harmonic oscillation. The slope of the wave face is $\cos(\omega' t)$.

The maximum is roll angle increases with amplitude, and it decreases with boat friction and with the radius of gyration. The boater seeking comfort should look for larger boat, lower seas, and more boat friction (keel, stabilizers, flopper-stoppers, etc.

Roll Mitigation Techniques

We've seen that roll is initiated by the torque generated by an arriving wave. The wave can arrive from any direction, but roll results from torque that is generated perpendicular to the vessel's longitudinal axis, i.e. it is a side-to-side motion. Motions in other directions, like yaw and pitch are not as readily mitigated because the boat's longitudinal axis is most unstable and, therefore, easiest to destabilize and to restabilize. All methods of stabilization rely on generation of an opposing torque: a wave arriving on the starboard beam generates a torque that creates a roll to the port, so stabilization require creation of a torqueing force toward the starboard.

Any method that creates counter-torque mitigates roll. The simplest method, observed on sailboats, is to have the vessel's occupants shift position so as to always sit on the side on which the wave arrives. In power boats a variety of passive methods are used. One ancient passive method is to store ballast below the vessel's center of gravity or to add a keel to the boat. These work by reducing the center of gravity, thus increasing the metacentric height and reducing the roll angle; keels also employ hydrodynamic properties—the keel resists roll by pushing against the water, this increasing the friction between the boat and the water.

Yet another inexpensive passive method employing hydrodynamics is the bilge keel, shown below. This is a rigid metal extension attached to the hull that resists roll by increasing boat-water friction. A similar device is the "flopper-stopper," a mechanical system used on many commercial fishing vessels and some recreational boats. A flopper-stopper is a weight attached to a line hung vertically from a frame above the boat's deck, one on each side; when roll mitigation is needed, the floppers are lowered into the water. A roll to the port then creates a counter-acting torque on the starboard side as the weight pulls the starboard side down.

A Bilge Keel

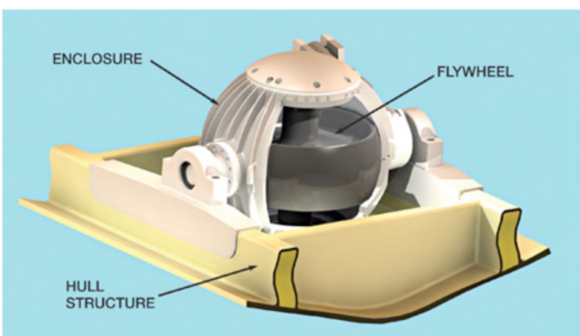
A Flopper-Stopper

Larger stabilized vessels use active methods involving gyroscopic sensing of the vessel's position and creation of antiroll actions. One such method is external fins: a gyroscope senses the boat's roll angle, sending messages to fins attached to each side to rotate them to offset the torqueing force of the arriving wave. Unlike a flopper stopper, these are effective even when vessels are underway, though as vessel speed increase the effectiveness declines because the fin positions can't adjust as quickly as needed; also, as speed increases and fin effectiveness drops, the stresses on the stabilizing fins increase, adding to wear.

External Fin Stabilization on a Private Yacht

A relatively new method (though it has been used since the early twentieth century) is internal gyroscopic stabilization. Gyroscopic roll stabilizers are not a new concept: the first U. S. patent was issued in 1904; the *USS Henderson*, a troopship launched in 1917, used gyroscopic stabilizers; and the Italian liner *Conte di Savoia*, launched in 1932, used several gyroscopic stabilizers—each of the liner's thee units each weighed over 100 tons. In 2004 Mitsubishi introduced its ARG (anti-roll gyroscope), and Seakeeper introduced an improved model in 2008.

As with external fins, a gyroscope senses the vessel's roll angle, but the information is used to create torque generated by a rapidly spinning flywheel installed within the hull. A simple cutaway of a Seakeeper device is shown below left; an installed device is shown below right.



Seakeeper Cutaway


Seakeeper Installed

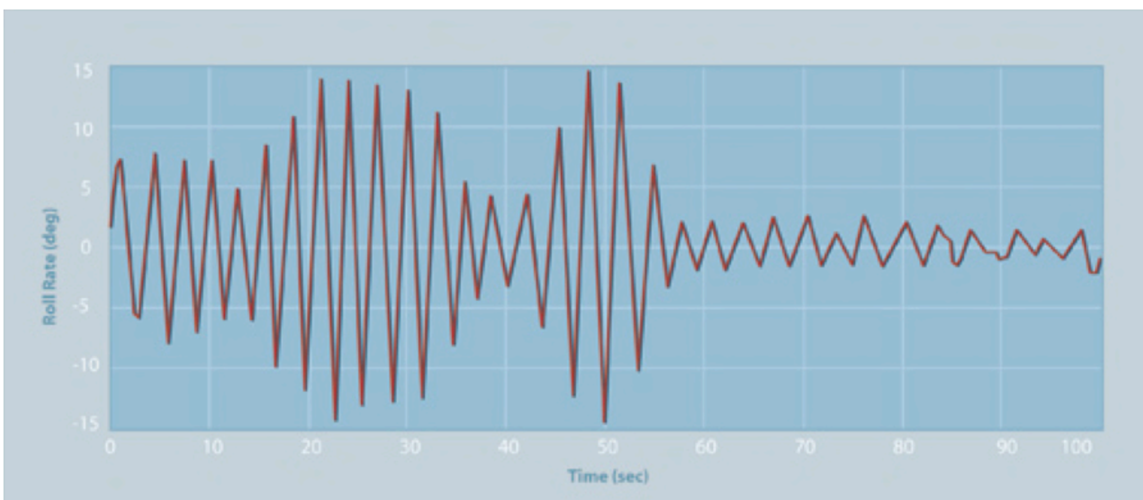
A flywheel is mounted on an axle that runs lateral to the boat's longitudinal axis. The flywheel is rotated at a high angular velocity, creating a torqueing force in the direction of the spin that increases with the flywheel's angular velocity and the flywheel weight.

The axle allows the device to tilt forward and aft, changing the direction of the torque: when the flywheel is tilted forward the torque is directed to one side of the vessel; when tilted aft, torque is directed to the other side. Sensors determine the state of the boat's roll and tell the device how to orient the flywheel to create a countering torque.

The gyroscopic stabilizer unit is installed within a metal globe with cooling fins around the outside. A near-vacuum is maintained within the globe to reduce air friction from the flywheel; this reduces the power required to run the system.

The largest Seakeeper unit, designed for a 100-foot yacht, spins at 10,000RPM and weighs about 10,000 pounds; larger yachts can require several units. It costs about $200,000 excluding installation and ancillary costs such as a generator is to provide the 3KW of power needed to power the unit.

In exchange, the gyroscopic stabilizer dramatically reduces roll, as shown in the chart below.



Roll Angle Effect Before and After Seakeeper Turned On

The gyro stabilization method uses a subtle property of spinning objects called *precession*--the the wobble seen when a toy top spins on its around its spin axis.

References

Denny, Mark. *Float Your Boat! The Evolution and Science of Sailing*, Johns Hopkins University Press, Baltimore, 2009.

Kimball, John. *The Physics of Sailing*, CRC Press, New York, 2010